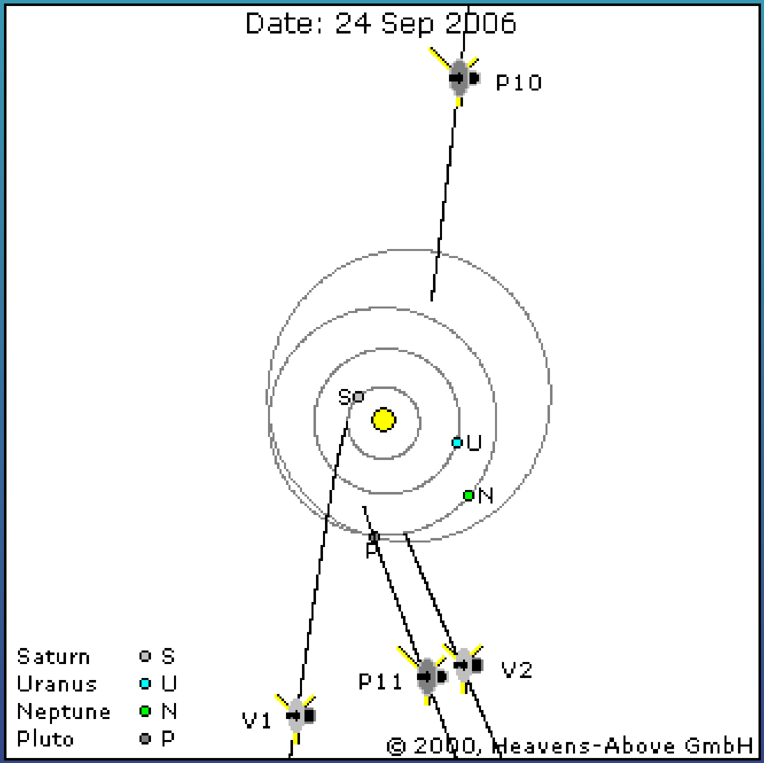
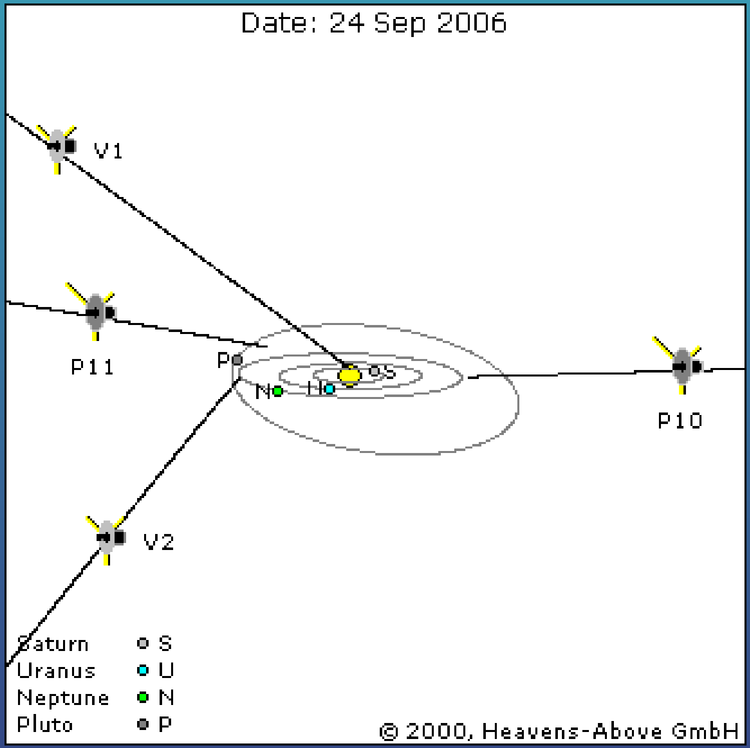
Space Propulsion

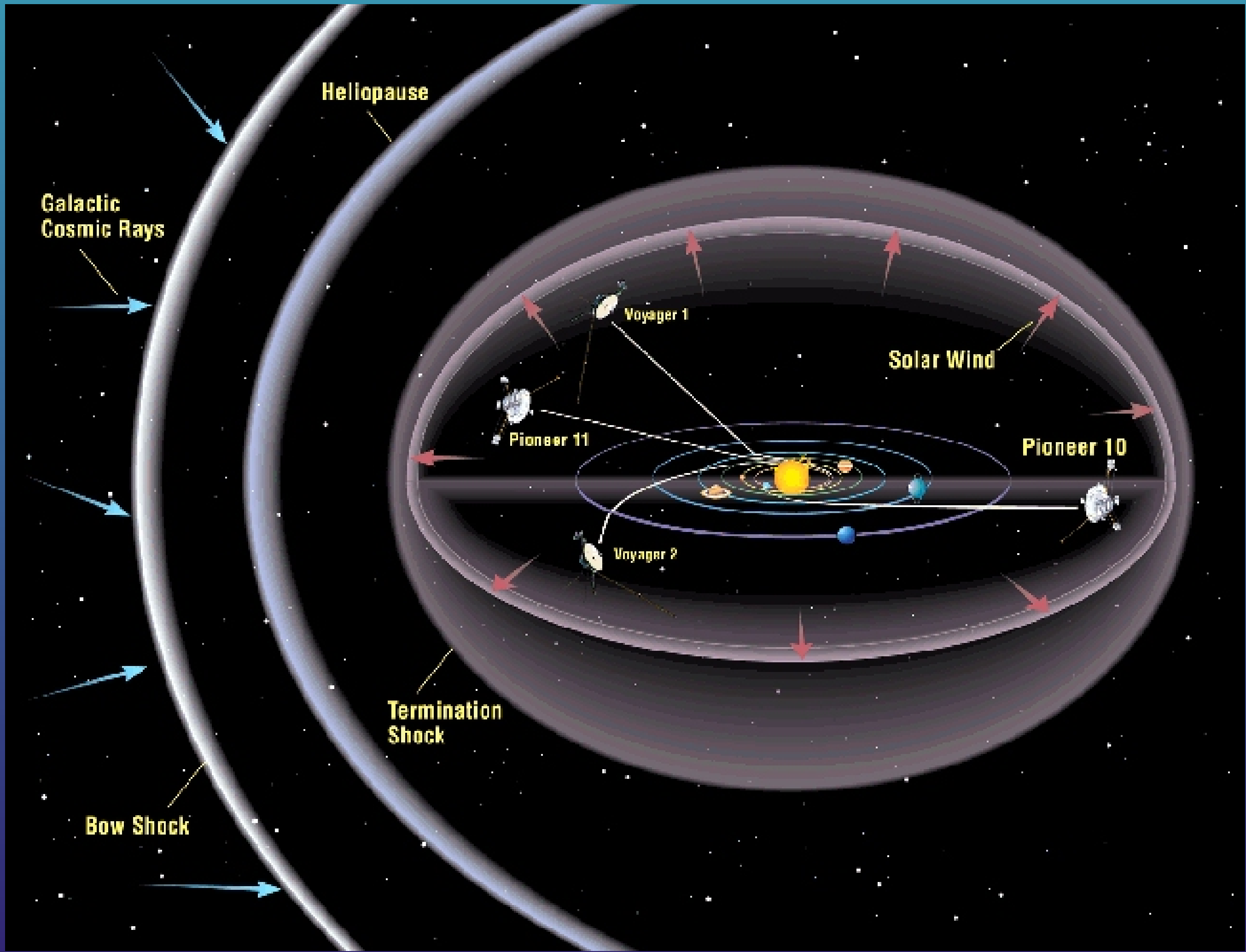
	Pioneer 10	Pioneer 11	Voyager 1	Voyager 2
Launch Date	Mar. 3,1972	Apr. 5,1973	Aug. 20,1977	Sept. 5,1977
Loss of Signal	2001 (1994 expected) (at 59 AU)	1996 (at 45 AU)	2012 (at 121 AU)	2013 (at 106 AU)
Departure velocity Asymptotic (AU/yr)	2.4	2.2	3.5	3.4
Trajectory Angle to Earth Orbit Plane (degrees	2.9	12.6	35.5	-47.5
Closest Stellar Approach				
Distance (ly)	3.27	1.65	1.64	0.80
Star	Ross 248	AC+793888	AC+793888	Sirius
Years to reach	32,600	42,400	40,300	497,000

Space Propulsion





Space Propulsion



Space Propulsion

Isaac Newton
1643 - 1727

Gottfried Wilhelm
Freiherr von Leibniz
1646 - 1716

1.	*Every body continues in its state of rest or of uniform motion in a straight line except insofar as it is compelled to change that state by an external impressed force*
2.	*The rate of change of momentum of the body is proportional to the impressed force and takes place in the direction in which the force acts.*
3.	*To every action there is an equal and opposite reaction*

$d\mathbf{p} / dt = \mathbf{F}$

Space Propulsion

Every particle of matter attracts every other particle of matter with a force directly proportional to the product of the masses and inversely proportional to the square of the distance between them.

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\vec{e}_r$$

$$U(r) = -G\frac{M}{r}$$

Space Propulsion

Johannes Kepler
1571 - 1630

Tycho Brahe
1546 - 1601

1.	***The planets move in ellipses with the sun at one focus***
2.	***Areas swept out by the radius vector from the sun to a planet in equal times are equal***
3.	***The square of the period of revolution is proportional to the cube of the semimajor axis.*** **That is,** $T^2 = const \times a^3$

Space Propulsion

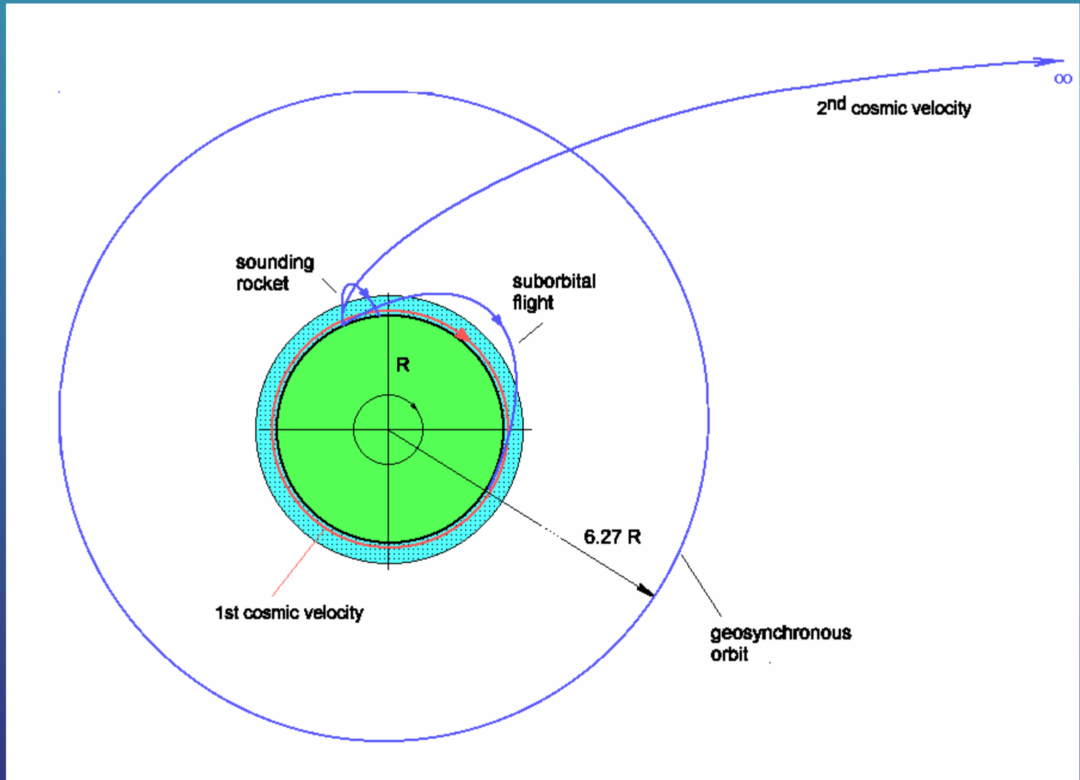
Circular orbit

$$\frac{mV^2}{r} = \frac{GmM}{r^2}$$

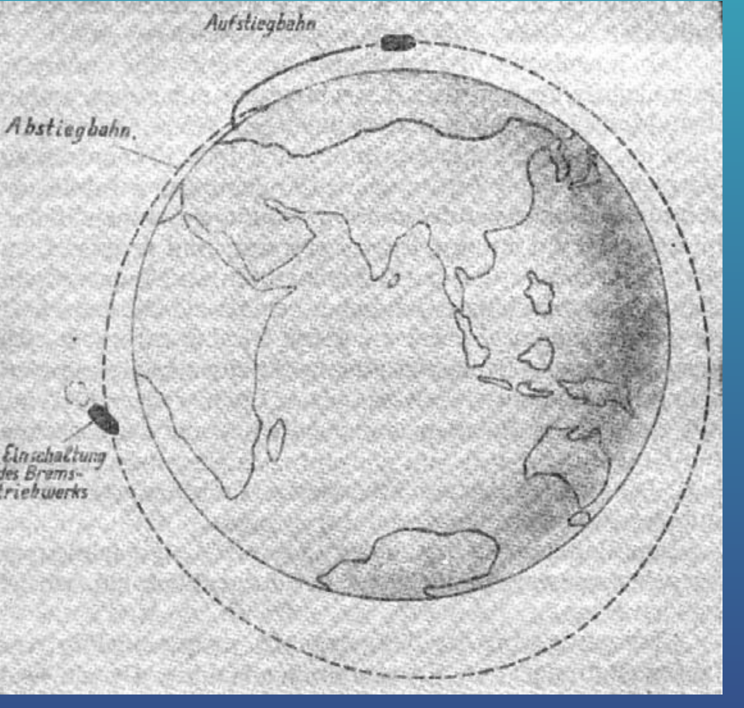
$$V = \sqrt{\frac{\mu}{r}}$$

$$P = \frac{2r\pi}{V} = 2\pi\sqrt{\frac{r^3}{\mu}}$$

Space Propulsion



Space Propulsion



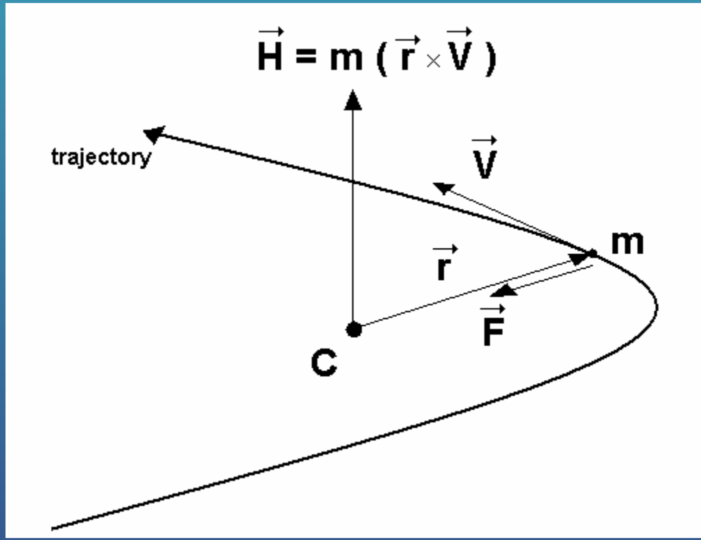
$$V_{orb} = \sqrt{\frac{\mu}{R}} \cong \sqrt{\frac{3.98x10^5}{6.73x10^3}} = 7.69 \quad [km/s]$$

$$V_{rot} = \frac{2\pi R}{24x3600} \cong 0.489 \quad [km/s]$$

$$V_1 = V_{orb} - V_{rot} \cong 7.20 \quad [km/s]$$

Space Propulsion

Gravitational trajectories



$$\vec{H} = \vec{r} \times m\vec{V}$$

angular momentum around point C

$$\frac{d\vec{H}}{dt} = m\left(\frac{d\vec{r}}{dt} \times \vec{V} + \vec{r} \times \frac{d\vec{V}}{dt}\right) = \vec{r} \times m\frac{d\vec{V}}{dt} = \vec{r} \times \vec{F} = \vec{M}$$

M … torque around C

Assumption:
Central force: F II r

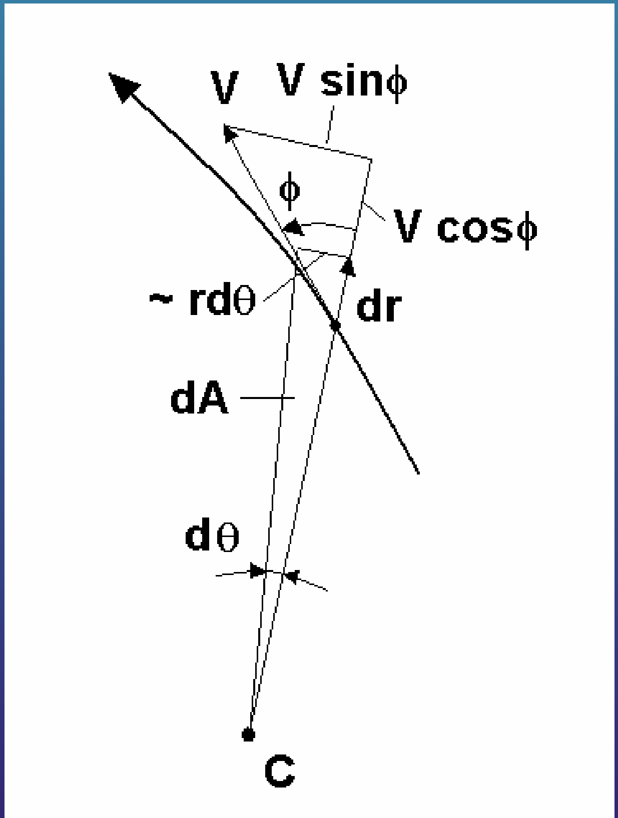
$$\vec{r} \times \vec{F} = \vec{M} = 0 \rightarrow \frac{d\vec{H}}{dt} = \vec{M} = 0 \rightarrow \vec{H} = m\vec{r} \times \vec{V} = const.$$

→ **trajectory remains in same plane perp. to H**

Gravitational trajectories

$$h = \left|\vec{H}/m\right| = \left|\vec{r}\times\vec{V}\right| = rV\sin\phi = const$$

specific angular momentum = const



$$d\theta = \left(V\sin\phi . dt\right)/r \rightarrow \frac{d\theta}{dt} = \frac{V\sin\phi}{r} = \frac{h}{r^2}$$

$$dA \cong \frac{1}{2}r.\left(rd\theta\right) = \frac{1}{2}rV\sin\phi . dt$$

$$\frac{dA}{dt} = \frac{1}{2}rV\sin\phi = \frac{h}{2} = const$$

→ 2nd **Kepler's law**: areal velocity is constant

Plane trajectories and constant areal velocity follow from central force requirement only; force field must not be $1/r^2$ and not even conservative

Space Propulsion

Gravitational trajectories

Assumption: conservative 1/r² force field

$$F = -\frac{dU}{dr} = -\frac{m\mu}{r^2}$$

$$\frac{V^2}{2} - \frac{\mu}{r} = \varepsilon = const.$$

conservation of total energy; ε is **specific total energy**;

$$V^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

magnitude² of velocity in polar coordinates (r, θ)

$$-\frac{\mu}{r} + \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\left(r\frac{d\theta}{dt}\right)^2 = \varepsilon$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\left(\frac{h}{r^2}\right)$$

$$\frac{d\theta}{dt} = \frac{V\sin\phi}{r} = \frac{h}{r^2}$$

$$\frac{h^2}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^2} - \frac{2\mu}{r} = \varepsilon$$

differential equ. of trajectory

$$d\theta = \frac{h/r^2}{\sqrt{\varepsilon + \frac{2\mu}{r} - \frac{h^2}{r^2}}}dr$$

subst. r=1/u

$$d\theta = -\frac{h}{\sqrt{\varepsilon + 2\mu.u - h^2u^2}}du$$

$$\theta + C = -h\int\frac{du}{\sqrt{\varepsilon + 2\mu.u - h^2u^2}}$$

general solution

Space Propulsion

Gravitational trajectories

$$\theta + C = -h\int \frac{du}{\sqrt{\varepsilon + 2\mu.u - h^2u^2}}$$

r = 1/u

$$r = \frac{h^2/\mu}{1 - \sqrt{1 + \frac{2\varepsilon.h^2}{\mu^2}}.\cos(\theta + C)}$$

when θ is counted from minimum r, then cos = -1

$$r = \frac{h^2/\mu}{1 + \sqrt{1 + \frac{2\varepsilon.h^2}{\mu^2}}.\cos\theta}$$

From geometry:

$$r = \frac{p}{1 + \varepsilon\cos\theta}$$

Is equation of conical section in polar coordinates (r,θ) when origin is in focal point; p is parameter and ε numerical excentricity of conic section;

ε > 1 …hyperbola

ε = 1 …parabola

ε < 1 …ellipse

ε = 0 … circle

Trajectories under influence of gravity of the sun are conical sections with the sun in one focal point

1st Kepler

Space Propulsion

Gravitational trajectories

$$r = \frac{h^2 / \mu}{1 + \sqrt{1 + \frac{2\varepsilon . h^2}{\mu^2}} . \cos\theta}$$

numerical excentricity $\underline{\varepsilon}$ of conical section

$\bar{\varepsilon} > 1 \rightarrow specific \quad energy \quad \varepsilon > 0 \rightarrow hyperbola$

$\bar{\varepsilon} = 1 \rightarrow specific \quad energy \quad \varepsilon = 0 \rightarrow parabola$

$\bar{\varepsilon} < 1 \rightarrow specific \quad energy \quad \varepsilon < 0 \rightarrow ellipse$

$$p = \frac{h^2}{\mu} \qquad \bar{\varepsilon} = \sqrt{1 + \frac{2\varepsilon . h^2}{\mu^2}}$$

from geometry

$$a = \frac{p}{1 - \bar{\varepsilon}^2} \rightarrow a = -\frac{\mu}{2\varepsilon}$$

parameter, semimajor axis and num. excentricity of trajectory follow from kinetic and dynamic parameters by analogy of anal. solution with geometry of conical sections

all trajectories with same semimajor axis have same (specific) total energy

Space Propulsion

$$\frac{dA}{dt}P = ab\pi = \pi a^2\sqrt{1-\bar{\varepsilon}^2} = \pi a^2\sqrt{\frac{p}{a}}$$

In case of closed trajectory (ellipse) product of constant areal velocity and period is equal to area of ellipse

from kinetics:
dA/dt = h/2

$$p = \frac{h^2}{\mu}$$

$$b = a\sqrt{1-\bar{\varepsilon}^2}$$

$$p = a\left(1-\bar{\varepsilon}\right)^2$$

from analyt. geometry

$$P^2 = \frac{4\pi^2 a^4}{h^2}\frac{p}{a} = \frac{4\pi^2 a^3}{\mu}$$

3rd Kepler
But also: period of elliptical trajectory only dependent on semimajor axis

Space Propulsion

$$r = \frac{h^2/\mu}{1+\sqrt{1+\frac{2\varepsilon.h^2}{\mu^2}}.\cos\theta}$$

$$a = -\frac{\mu}{2\varepsilon}$$

$$\bar{\varepsilon} = \sqrt{1+\frac{2\varepsilon.h^2}{\mu^2}}$$

$$p = h^2/\mu$$

geometric parameters of orbit can be derived from kinetic parameters of motion

$$specific \quad energy \quad \varepsilon > 0 \rightarrow hyperbola$$

$$specific \quad energy \quad \varepsilon = 0 \rightarrow parabola$$

$$specific \quad energy \quad \varepsilon < 0 \rightarrow ellipse$$

type of conics dependent on total energy

The orbit of a body is completely determined, when we know at a given point

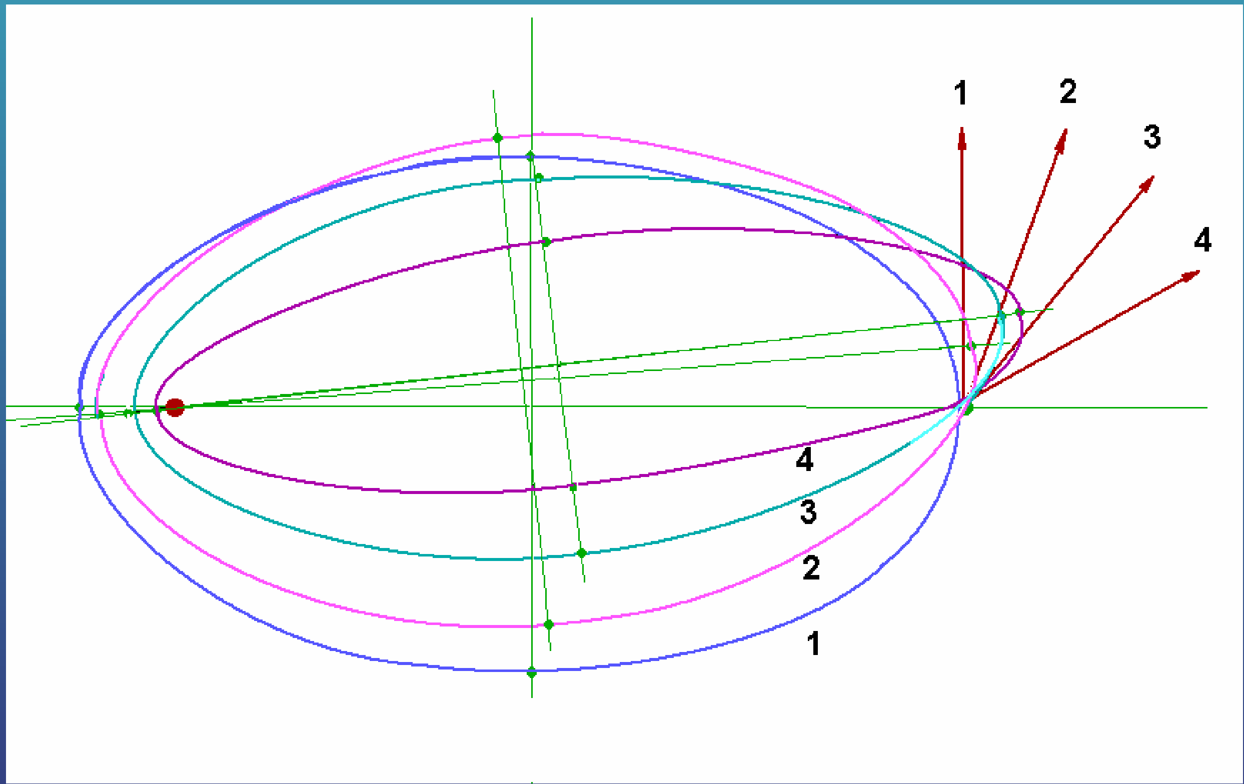
- the radius – vector from the central body
- the velocity vector

Or, equivalently r, V and included angle α

Space Propulsion

1	Evaluate $\mu = GM_{Sun}$	
2	The total energy per mass of the orbit is constant so by evaluating the kinetic and gravitational potentialenergy at one point in the orbit **(EQ 10)** we obtain	$\frac{E}{m} = \frac{V^2}{2} - \frac{GM_{sun}}{r}$
3	The energy per mass of the spacecraft determines the orbits semi-major axis **(EQ 11)**:	$a = -\frac{GM_{sun}}{2(E/m)}$
4	This then gives the circular velocity of the orbit **(EQ 3)**	$v_c = \sqrt{\frac{GM_{sun}}{a}}$
5	The period of the orbit is given by Kepler's third law:	$P = P_{earth}\left(\frac{a}{a_{earth}}\right)^{3/2}$
6	The areal velocity is know from the initial conditions (velocity and position) of the spacecraft; α being the angle between radiusvector and S/C direction	$A = \frac{1}{2} r.V.\sin\alpha$
7	The other method of determining areal velocity gives us the eccentricity of the orbit, by taking the ellipse area as Aell = πa2(1-e2)1/2	$A = \frac{A_{ell}}{P} = \frac{\pi a^2\sqrt{1-\varepsilon^2}}{P}$ $\varepsilon = \sqrt{1-\left(\frac{AP}{a^2\pi}\right)^2}$
8	We now know the size and shape of the orbit and can determine the extent of the orbit from **(EQ 16)** and **(EQ 18)**	$r_p = a(1-\varepsilon)$ $r_a = a(1+\varepsilon)$
9	The final parameter is the true anomaly as determined by the angle the craft is from perihelion of the new orbit (see ellipse equation in Section 2.3.1)	$\cos\Theta = \frac{1}{\varepsilon}\left(\frac{a}{r}(1-\varepsilon^2)-1\right)$

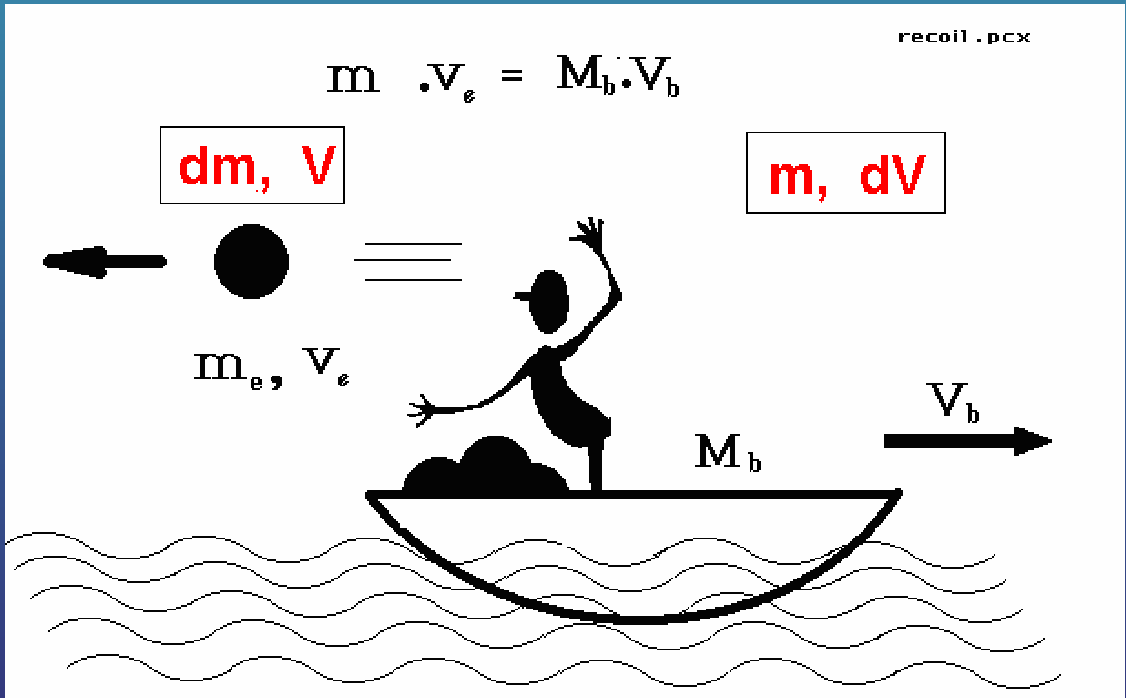
Space Propulsion



Elliptical orbits passing through same point with identical velocities into different directions

Space Propulsion

Reaction propulsion



momentum conservation

$$mV = const.$$

$$dp = d(mV) = mdV + Vdm = 0$$

$$dV = -V_e \frac{dm}{m}$$

$$\int dV = -V_e \int \frac{dm}{m}$$

$$\Delta V = -V_e \ln \frac{m_i}{m_f}$$

Tsiolkovski equation

Space Propulsion

Tsiolkovsky equation

$$\Delta V = -V_e \ln \frac{m_i}{m_f}$$

$$\frac{m_i}{m_f} = e^{\frac{\Delta V}{-v_e}}$$

since direction of v_e (exhaust velocity) is opposite to velocity gain ΔV, the ratio $-\Delta V/v_e$ is always positive; therefore we can express the exponent as $\Delta V/|v_e|$

$$\frac{m_i}{m_f} = e^{\frac{\Delta V}{|v|_e}}$$

- initial mass increases exponentially with ΔV (@ m_f = const.)
- decreases exponentially with v_e
- final mass, which can be brought into orbit with ΔV decreases with increasing ΔV and increases with v_e

Space Propulsion

Thrust

Thrust is the force propelling a rocket; it is the reaction force to the force accelerating the exhaust particles. We consider the exhaust consisting of N identical particles (gas, ions, electrons, stones,…) of mass m

$$T = \frac{dP}{dt} = \frac{d}{dt}\left(N.p\right) = \frac{dN}{dt}.p = \left(\dot{N}\, m\right)V_e = \dot{m}.V_e$$

$\dot{m}$	mass flow [kg/s]
V_e	exhaust velocity [m/s]
T	thrust [N]

Space Propulsion

Total impulse

Total impulse is the total momentum gained during the burn time t_b of a thruster

definition

$$I = \int_0^{tb} T dt \quad [N.s]$$

When thrust is constant over time, or at least during thruster – on time intervals, total impulse can be written as

$$I = T.\tau$$

$$I = \int_0^{\tau} T dt = \int_0^{\tau} \left(\frac{dm}{dt} \right) V_e dt = V_e \int_0^{\tau} dm = V_e . m_p$$

m_p … propellant mass used during mission time τ

V_e … exhaust velocity, assumed to be constant during mission

Space Propulsion

Specific impulse

$$I_{sp} = \frac{dp}{dm}$$

definition

what is the momentum produced per unit of mass expelled**?**

The higher this ratio, the higher is the velocity gain of a rocket upon exhaustion of ist fuel mass;. **I_{sp} is an important quality parameter**

$$I_{sp} = \frac{dp/dt}{dm/dt} = \frac{\dot{p}}{\dot{m}} = \frac{T}{\dot{m}} \quad [m/s]$$

$\dot{m}$ mass flow [kg/s]

$$I_{sp} = \frac{dp}{dm} = \frac{d(m.V_e)}{dm} = V_e \quad [m/s]$$

V_e exhaust velocity, assumed to be constant

Space Propulsion

jet power

definition

$$P_j = \frac{dE_j}{dt} = \dot{N}\frac{mV_e^2}{2} \quad [W]$$

jet power is the kinetic energy, emitted per time unit from a S/C

$$P_j = \frac{\left(\dot{N}\, mV_e\right)V_e}{2} = \frac{TI_{sp}}{2}$$

Space Propulsion

specific power

$$P_{sp} = \frac{P_{jet}}{T} \quad [W/N]$$

specific power is the beam power P_{jet}, necessary to produce a unit of thrust

$$P_{sp} = \frac{\dot{m} V_e^2 / 2}{\dot{m} V_e} = \frac{V_e}{2} \quad [m/s],[W/N]$$

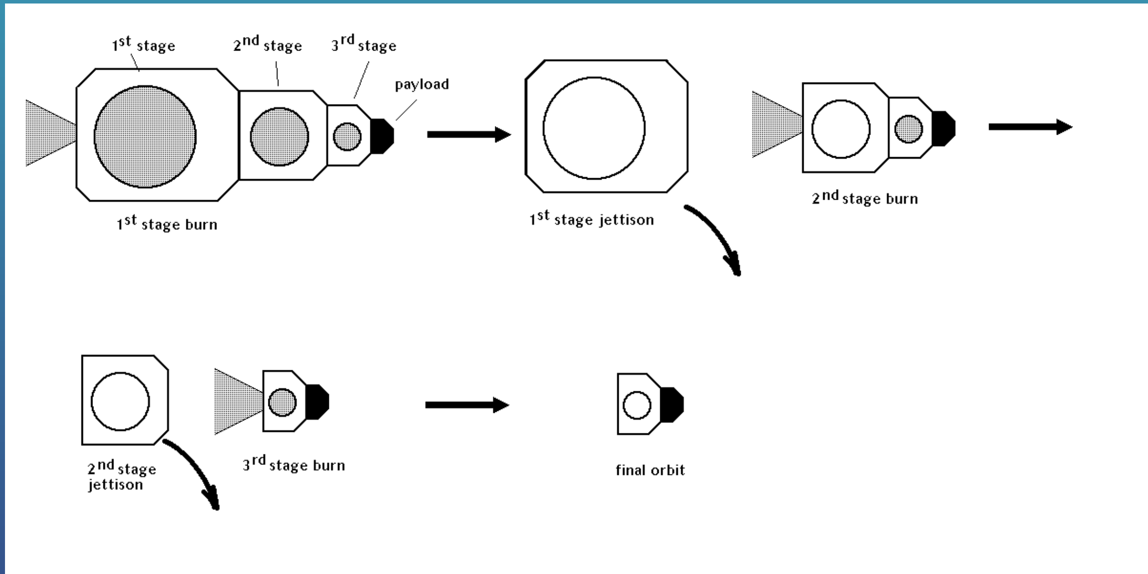
Space Propulsion

these purely mechanical relationships are valid independent of the methods used to accelerate exhaust particles

$T = \dot{m} . I_{sp}$	[N]	thrust
$I = \int_0^\tau T dt = I_{sp} m_p \equiv T\tau$	[N.s]	total impulse
$P_j = \frac{dE_j}{dt} = \frac{T I_{sp}}{2}$	[W]	jet power
$I_{sp} = \frac{dp}{dm} = V_e = \frac{T}{\dot{m}}$	[m/s]	specific impulse
$P_{sp} = \frac{P_j}{T} = \frac{I_{sp}}{2}$	[W/N], [m/s]	specific power
$m_i / m_f = e^{\frac{\Delta V}{Isp}}$ $\Delta V = I_{sp} \ln\left(m_i / m_f\right)$	[1]	Tsiolkovsky equ. (rocket equ.)

Space Propulsion

The staging principle



$$R_j = \left(m_i / m_f\right)_j$$

initial / final mass ratio of j^th stage

$$\Delta V_1 = I_{sp} \ln R_1$$

$$\Delta V_2 = I_{sp} \ln R_2$$

....

$$\Delta V_n = I_{sp} \ln R_n$$

velocity gains of individual stages

$$\Delta V = I_{sp} \ln(R_1.R_2...R_n)$$

total velocity gain (of final stage)

Space Propulsion

The staging principle

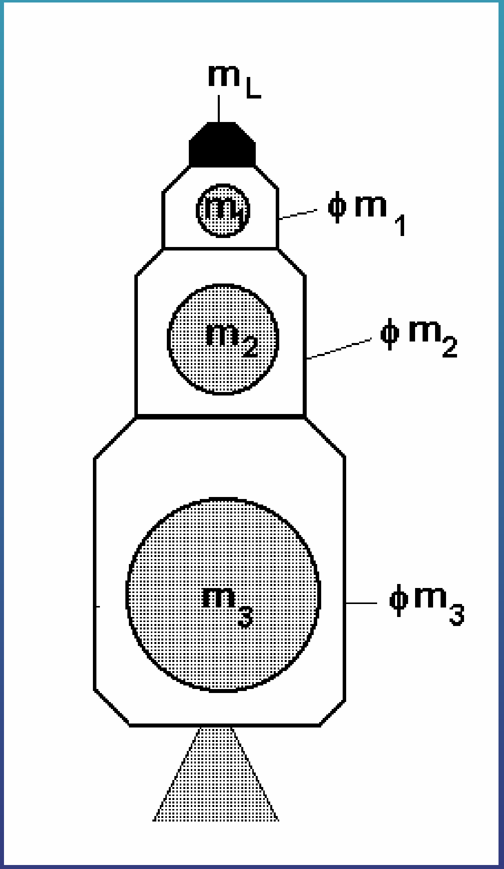
when the rocket motors of all stages have the same specific impulse I_{sp}, the velocity difference of the final stage with respect to the initial velocity is

$$\Delta V = I_{sp} \ln(R_1.R_2...R_n)$$

when the mass ratios of all stages are identical (R_j = R)

$$\Delta V = I_{sp} \ln(R^n) = n.I_{sp}.\ln R$$

Space Propulsion



- **a fixed total mass M of propellant is available for acceleration of a payload of mass m_L**
- **compare the velocity gains, when the propellant is consumed in a single – stage or a multi – stage rocket**

Assumptions:
- initial / final mass ratios identical = R for all stages
- mass of supporting structure in each stage is same fraction ϕ of propellant mass of respective stage (ϕ = „tankage factor“)

1. St. $$R = \frac{m_L + m_1(1+\phi)}{m_L + \phi m_1}$$ $$m_1 = \frac{R-1}{1-\phi(R-1)} m_L$$

2. St. $$R = \frac{m_L + (m_1+m_2)(1+\phi)}{m_L + m_1(1+\phi) + \phi m_2}$$ $$m_2 = \frac{R-1}{1-\phi(R-1)}\left[m_L + (1+\phi)m_1\right]$$

3. St. $$R = \frac{m_L + (m_1+m_2+m_3)}{m_L + (m_1+m_2)(1+\phi) + \phi m_3}$$ $$m_3 = \frac{R-1}{1-\phi(R-1)}\left[m_L + (1+\phi)(m_1+m_2)\right]$$

$$\rho = \frac{R-1}{1-\phi(R-1)}$$

$$\psi = (1+\phi)\rho$$

$$m_i = \rho m_L + \psi . S_{i-1}$$

propellant mass for i[th] stage;
S_i … sum of propellant masses m_1, m_2, …, m_i

Space Propulsion

$$m_i = \rho m_L + \psi . S_{i-1}$$

$$S_1 = \rho m_L$$
$$S_2 = m_2 + S_1 = \rho m_L + \psi S_1 + S_1 = \rho m_L + (1+\psi) S_1 = \rho m_L [1 + (1+\psi)]$$
$$S_3 = m_3 + S_2 = \rho m_L + \psi S_2 + S_2 = \rho m_L + (1+\psi) S_2 = \rho m_L \left[1 + (1+\psi) + (1+\psi)^2\right]$$
$$S_4 = m_4 + S_3 = \rho m_L + \psi S_3 + S_3 = \rho m_L + (1+\psi) S_3 = \rho m_L \left[1 + (1+\psi) + (1+\psi)^2 + (1+\psi)^3\right]$$

$$S_n = \rho m_L \sum_0^{n-1} (1+\psi)^i = \rho m_L \frac{(1+\psi)^n - 1}{\psi} = m_L \frac{(1+\psi)^n - 1}{1+\phi}$$

total propellant mass for n stages with equal tankage factor ϕ and equal initial / final mass ratio R

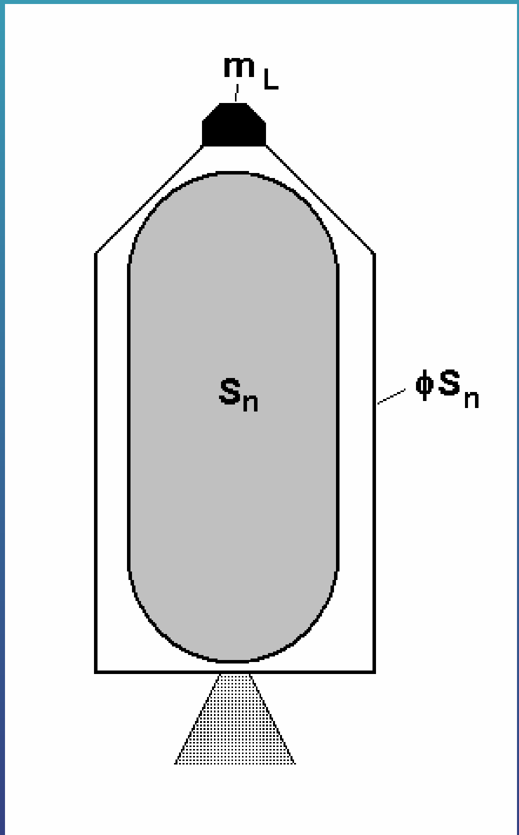
In an n – stage rocket, velocity gain in each stage is

$$\Delta V_i = I_{sp} \ln R$$

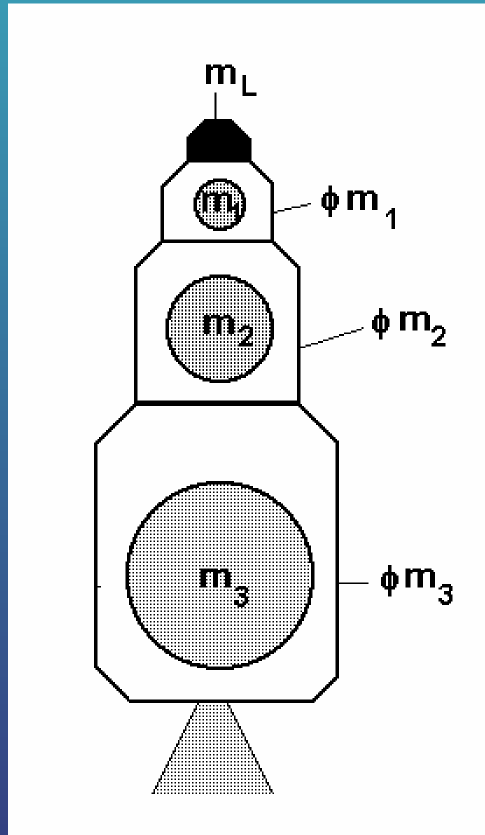
and total velocity gain of n stages is

$$\Delta V = n \Delta V_i = n I_{sp} \ln R$$

Space Propulsion



Single- and multi – stage rockets using the same amount of propellant to accelerate same payload



$$\Delta V = I_{sp}.\ln\left[\frac{(1+\phi)S_n + m_L}{m_L + \phi S_n}\right] = $$

$$= I_{sp}.\ln\left[\frac{1+\phi}{\phi + (1+\psi)^{-n}}\right]$$

$$\Delta V = n.I_{sp}.\ln R$$

$$S_n = m_L \frac{(1+\psi)^n - 1}{1+\phi}$$

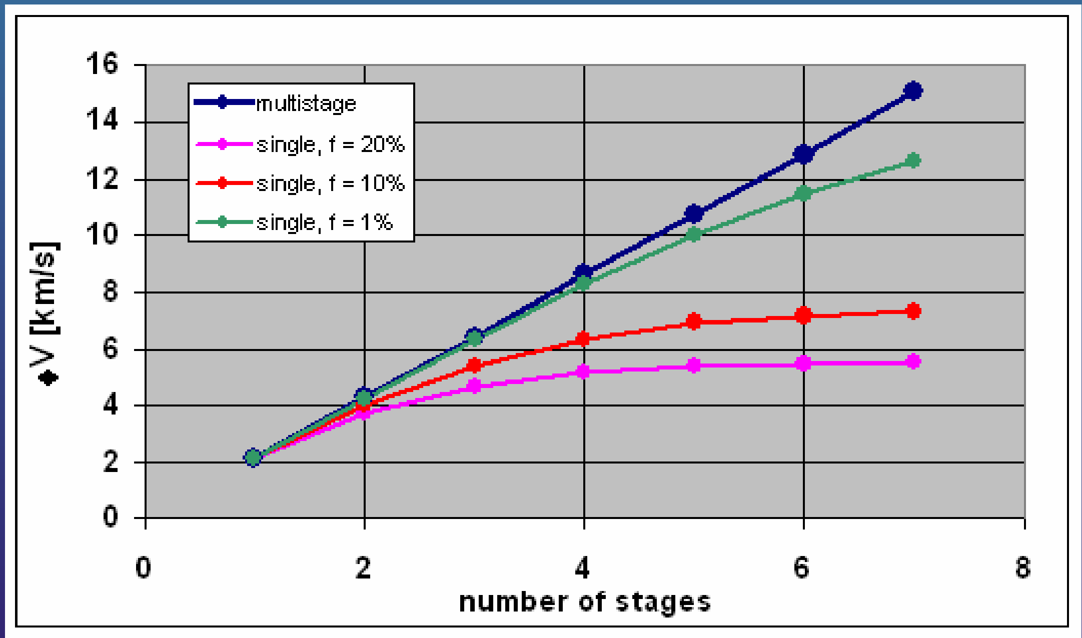
Space Propulsion

Check
for tankage factor $\phi \rightarrow 0$, we have rockets consisting of payload and fuel only and single- and „multistage“ rockets with same payload and fuel masses must have the same ΔV

$$\rho = \frac{R-1}{1-\phi(R-1)} \rightarrow R-1$$

$$\psi = (1+\phi)\rho \rightarrow R-1$$

$$\Delta V_{single} = I_{sp}.\ln\left[\frac{1+\phi}{\phi+(1+\psi)^{-n}}\right] \rightarrow I_{sp}\ln\left[\frac{1}{R^{-n}}\right] = nI_{sp}\ln(R) = \Delta V_{multi}$$



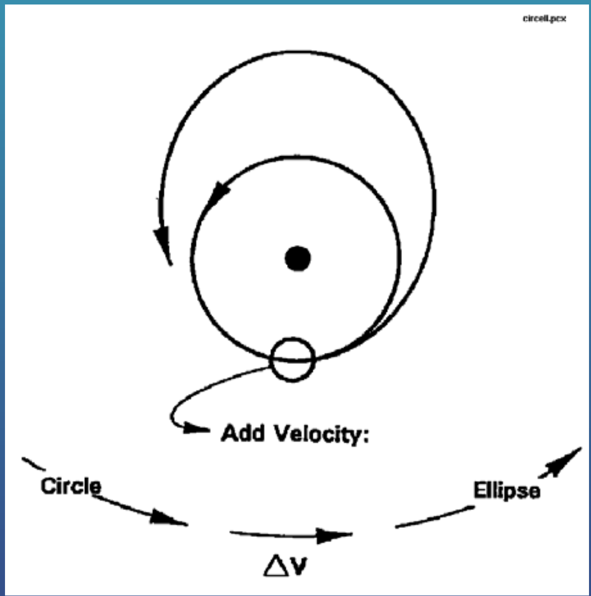
Space Propulsion

Mission Design and Attitude Control

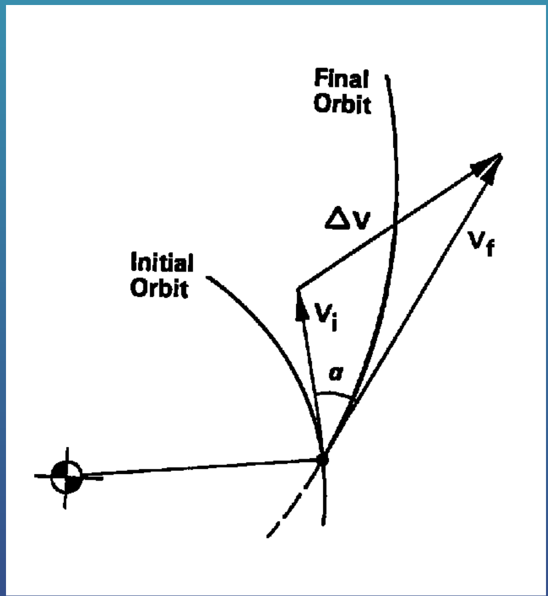
Task	Description
Mission design	(Translational velocity change)
Orbit changes	Convert one orbit to another
Plane changes	Change orbital plane, other orbit parameters remaining constant
Orbit trim	Remove launch vehicle errors
Stationkeeping	Maintain constellation position
Repositioning	Change constellation position
Attitude Control	(Rotational velocity change)
Thrust vector control	Remove vector errors
Attitude control	Maintain an attitude
Attitude changes	Change attitudes
Reaction wheel unloading	Remove stored momentum
Maneuvering	Repositioning the spacecraft axes

Space Propulsion

coplanar orbit changes



changing a circular orbit to a coplanar elliptical orbit



generalised coplanar maneuvre

$$\Delta V^2 = V_i^2 + V_f^2 - 2V_iV_f\cos\alpha$$

ΔV is smallest when this term is largest ➔ cosα = 1

- the transfer can be made at any intersection of two orbits.
- the least velocity change is necessary when the orbits are tangent and α is zero

Space Propulsion

Fuel consumption for orbital maneuvre with total velocity change ΔV

Tsiolkovsky:

$$m_i / m_f = e^{\frac{\Delta V}{Isp}}$$

required fuel mass:

$$m_p = m_i - m_f = m_i \left[1 - \exp\left(-\frac{\Delta V}{I_{sp}} \right) \right]$$

$$m_p = m_f \left[\exp\left(\frac{\Delta V}{I_{sp}} \right) - 1 \right]$$

Space Propulsion

Example 1: Simple Coplanar Orbit Change

Consider an initially circular low Earth orbit at 300-km altitude. What **velocity increase** would be required to produce an elliptical orbit 300 x 3000 km in altitude? What would be the **fuel consumption** for a 750 kg (empty) S/C if I_{sp} = 3100 m/s ?

The gravity parameter of Earth is μ=398,600.4 km³/s² Radius of Earth is ≈ R = *6387 km*

velocity on initial circular orbit:

$$V = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398{,}600.4}{(300 + 6378.14)}} = 7.726 \quad km/s$$

semimajor axis of final elliptical orbit:

$$a = \frac{r_a + r_p}{2} = \frac{(300+6378) + (3000+6378)}{2} = 8028 \quad [km]$$

velocity at periapsis of final orbit:

$$V_p = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} = \sqrt{\frac{2(398{,}600)}{6678} - \frac{398{,}600}{8028}} = 8.350 \quad km/s$$

velocity change =

$$\Delta V = V_p - V = 8.350 - 7.726 = 0.624 \quad km/s$$

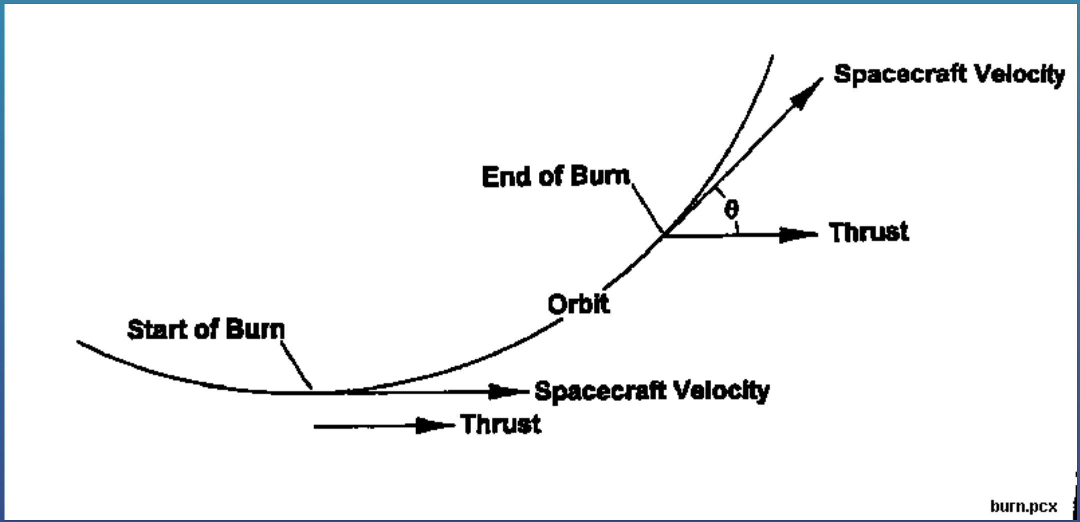
fuel consumption

$$m_p = m_f \left[\exp\left(\frac{\Delta V}{I_{sp}} \right) - 1 \right] = 750 \left\{ \exp\left[\frac{(624)}{(3100)} \right] - 1 \right\} = 167.2 \quad kg$$

Velocity changes, made at periapsis, change apoapsis radius but not periapsis radius, and vice versa; the radius at which the velocity is changed remains unchanged. As you would expect, the plane of the orbit in inertial space does not change as velocity along the orbit is changed. Orbital changes are a reversible process.

Space Propulsion

finite burn losses

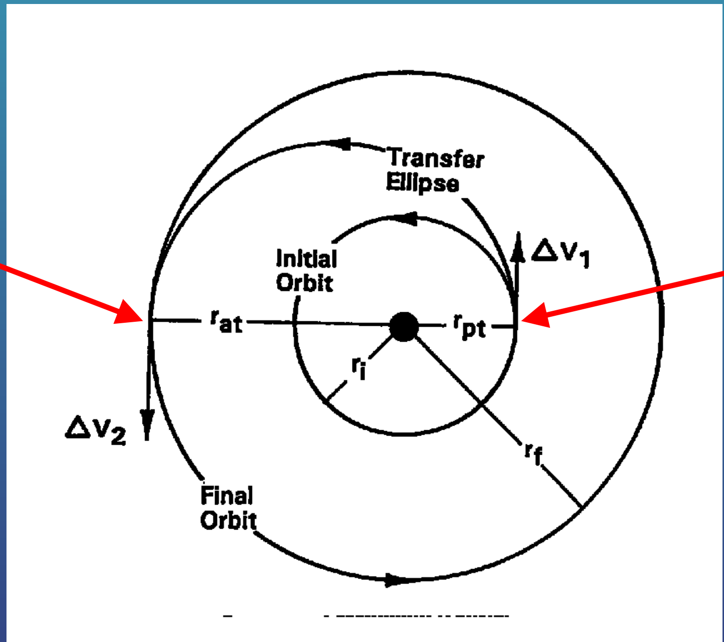


- thrust vector is held inertially fixed during the burn
- orbital elements change continuously during burn
- angle between thrust and velocity increases during burn
- at constant thrust, acceleration in flight direction decreases during burn

Space Propulsion

Hohmann transfer: minimum energy transfer between circular orbits

orbit circularisation



transfer orbit insertion

$$V = \sqrt{\frac{\mu}{r}}$$

$r_f > r_i \rightarrow V_f < V_i$

nevertheless all maneuvers are **accelerating**

transfer orbit:

periapsis radius = radius of initial orbit
apoapsis radius = radius of final orbit

Space Propulsion

Example 3: Hohman transfer from circular Earth orbit (altitude = 200 km) to geostationary orbit (r = 42219 km); what is fuel consumption to bring a 1 t payload to GEO with a specific impulse of 3100 [m/s]?

Velocity in LEO:

Velocity in LEO:

$$V = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398{,}600}{6387+200}} = 7.78 \quad [km/s]$$

Velocity in GEO similarly is

3.07 [km/s]

Semimajor axis of transfer ellipse is

$$a = \frac{(6387+200)+42219}{2} = 24403 \quad [km]$$

Perigee velocity in transfer ellipse is:

$$V_p = \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}} = \sqrt{\frac{2*398600}{6387+200} - \frac{398600}{24403}} =$$

$$= 10.22 \quad [km/s]$$

Space Propulsion

Velocity increase in transfer orbit insertion:

$$\Delta V_i = 10.22 - 7.78 = 2.44 \quad [km/s]$$

Apogee velocity in transfer ellipse is

$$V_a = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}} = \sqrt{\frac{2*398600}{42219} - \frac{398600}{24403}} = 1.60 \quad [km/s]$$

Velocity increase at circularization:

$$\Delta V_{circ} = 3.07 - 1.60 = 1.47 \quad [km/s]$$

Adding up to a total velocity increase of

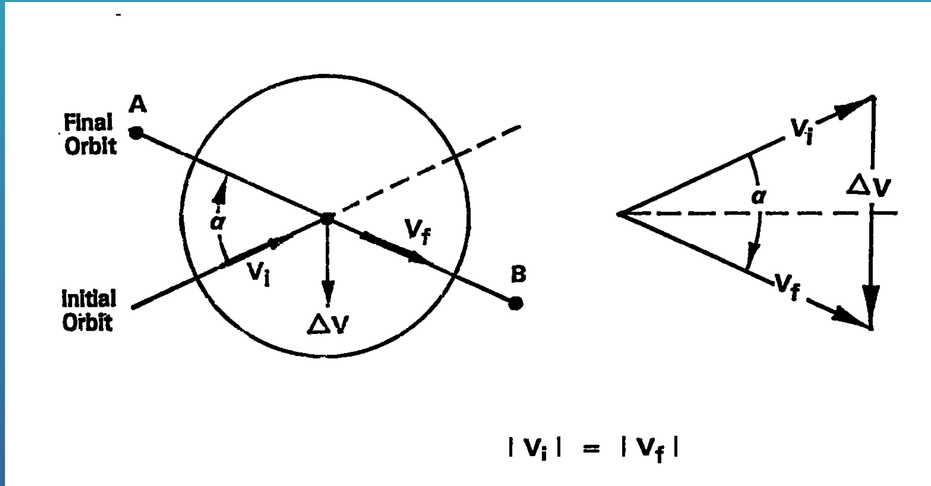
$$\Delta V_{tot} = 2.44 + 1.47 = 3.91 \quad [km/s]$$

Fuel consumption is:

$$m_p = m_f \left[\exp\left(\frac{\Delta V}{I_{sp}}\right) - 1 \right] = 1000 \left\{ \exp\left[\frac{(3910)}{(3100)}\right] - 1 \right\} = 2530 \quad [kg]$$

The efficiency of the Hohmann transfer comes from the fact that the two velocity changes are made at points of tangency between the trajectories.

Space Propulsion



plane change maneuver

$$\Delta V = 2V_i \sin\frac{\alpha}{2}$$

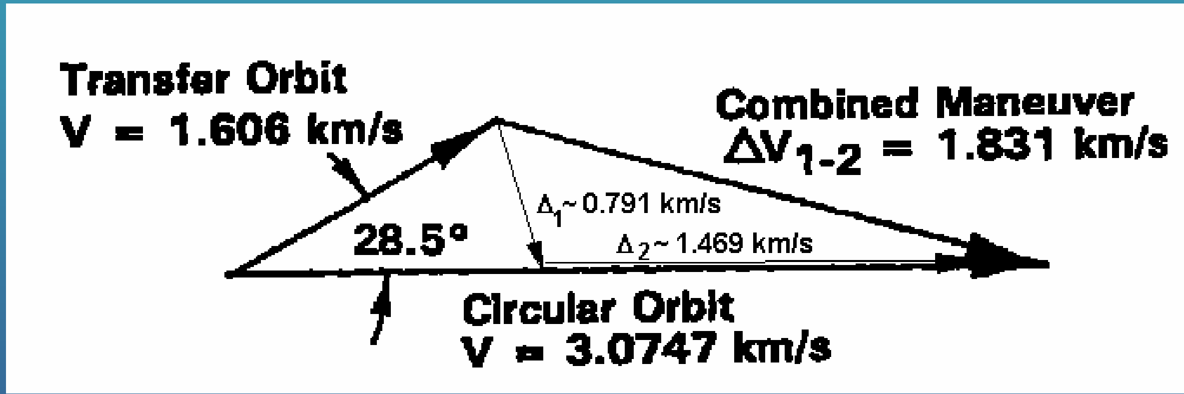
without velocity change

Plane changes are expensive on a propellant basis.
A 10-deg plane change in low Earth orbit would.require a velocity change of about 1.4 km/s.
For a 500 kg spacecraft, this plane change would require **292 kg** of propellant, if one assumes an I_{sp} of 3100 m/s

The equation shows that it is important to change planes through the smallest possible angle and at the lowest possible velocity.

The lowest possible velocity occurs at the longest radius, that is, **at apoapsis.**

Space Propulsion

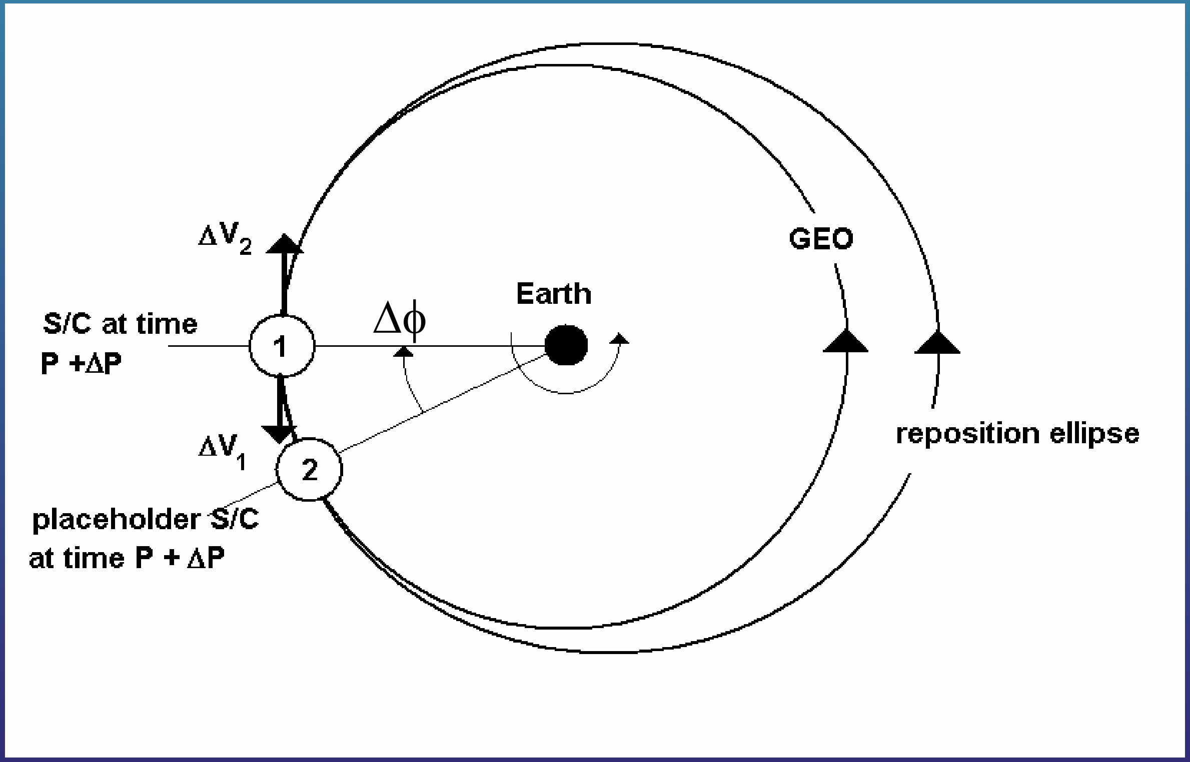


Combined maneuver: $\Delta V_{1\text{-}2} = 1.831\ km/s$

For separate maneuvers,
plane change maneuver: $\Delta V_1 = 0.791\ km/s$;
circularization maneuver: $\Delta V_2 = 1.469\ km/s$; total $\Delta V = 2.260\ km/s$.

Example 5: Repositioning

Consider a geosynchronous 1t spacecraft that is required to reposition by 2-deg, counter to the velocity vector (westward), in a maneuvering time of one sidereal day (one orbit). What is the fuel consumption for that maneuver, assuming an Isp of 3100 m/s?



S/C moves in reposition ellipse
„placeholder“ S/C orbits in GEO

Space Propulsion

The elements of a geosynchronous orbit are

r = 42,164.17 km (circular)

P = 86,164.09 s

V = 3.07466 km/s

ΔP is equal to the time required for 2 deg of motion on a geosynchronous orbit

$$\Delta P = \frac{\Delta\varphi^0}{360} P = \frac{2 * 86{,}164.09}{360} = 478.689 \quad [s]$$

The period for the spacecraft on the elliptical reposition orbit is

P = 86,164.09 + 478.689 = 86,642.78 [s]

semimajor axis of the reposition orbit

$$a = \sqrt[3]{\frac{P^2 \mu}{4\pi^2}} = \sqrt[3]{\frac{(86{,}642.78)^2 (398{,}600)}{4\pi^2}} = 42{,}320 \quad [km]$$

velocity at periapsis of an elliptical orbit with semimajor axis ***a***

$$V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} = \sqrt{\frac{2(398{,}600)}{42164} - \frac{398{,}600}{42320}} = 3.08032 \quad [km/s]$$

velocity change to place the spacecraft on the reposition ellipse

3.08032km/s - 3.07466km/s = 5.66 m/s

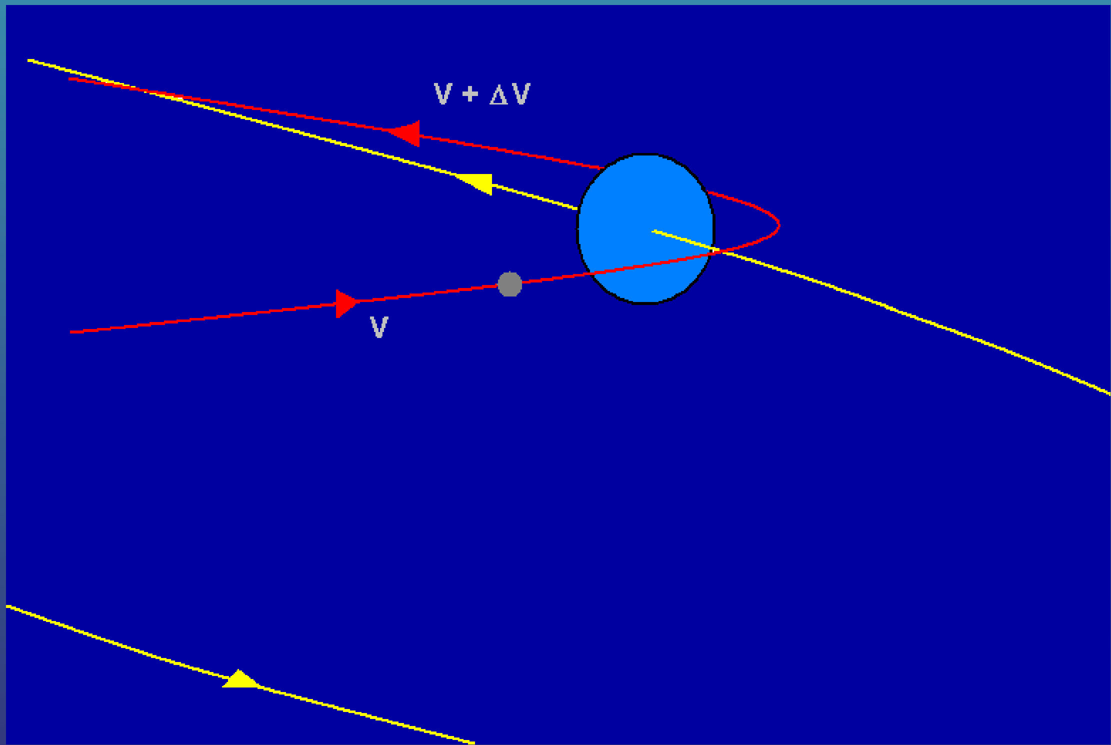
The same velocity change (in the opposite direction is necessary for recirculation of repositioning orbit

Propellant consumption under assumption of negligible mass change

$$m_p = m_f \left[\exp\left(\frac{\Delta V}{I_{sp}} \right) - 1 \right] = 1000 \left\{ \exp\left[\frac{11.32}{3100} \right] - 1 \right\} = 3.66 \quad [kg]$$

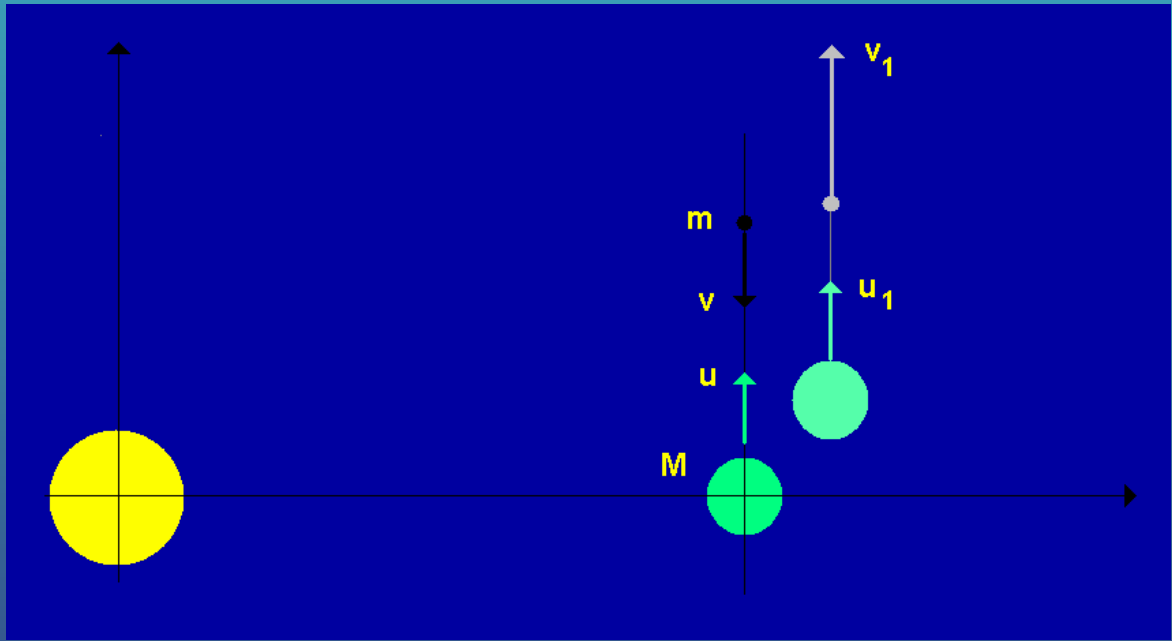
Space Propulsion

Gravity assist maneuvre (slingshot)



S/C may gain velocity in the sun – fixed system when passing close to a planet

Space Propulsion



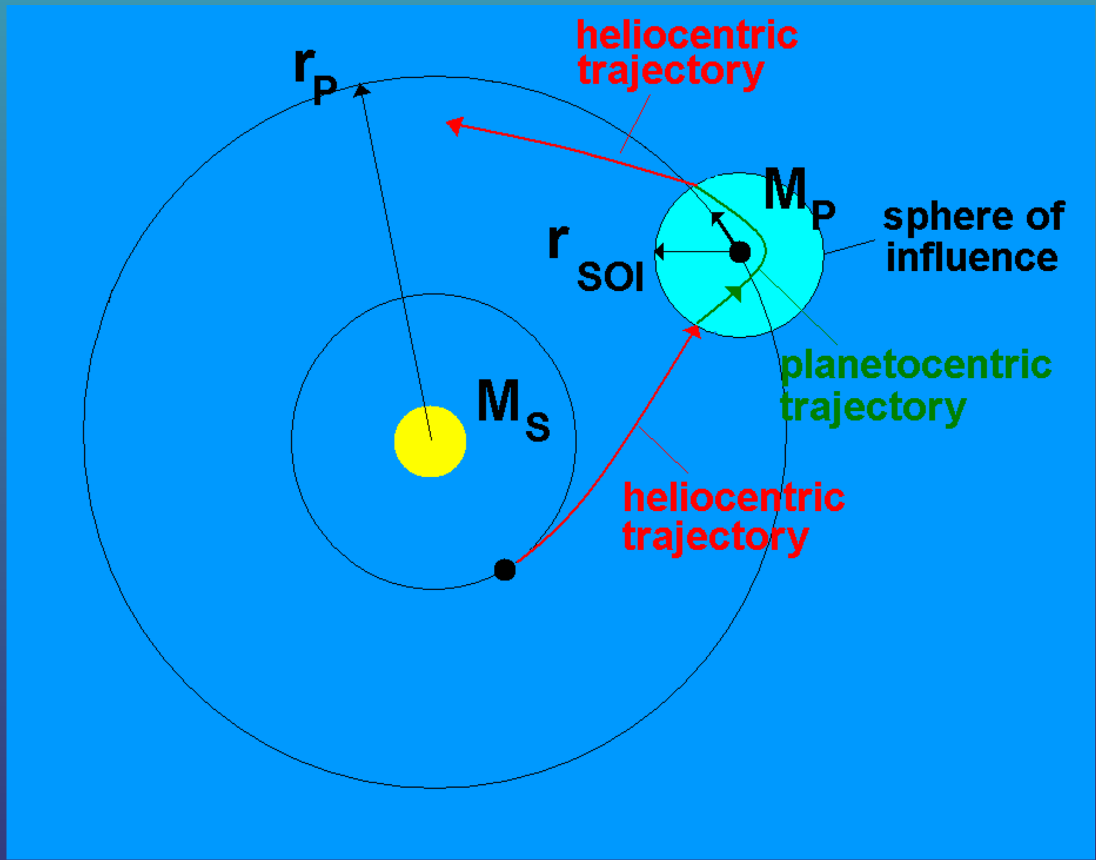
Basically a 3 – body problem; approximation possible, when m <<M << M_{sun}
Consider head – on elastic collision in a fixed coordinate system

$$Mu^2 + mv^2 = Mu_1^2 + mv_1^2$$
$$Mu - mv = Mu_1 + mv_1$$

$$v_1 = \frac{(1-\mu)v + 2u}{1+\mu} \cong v + 2u$$

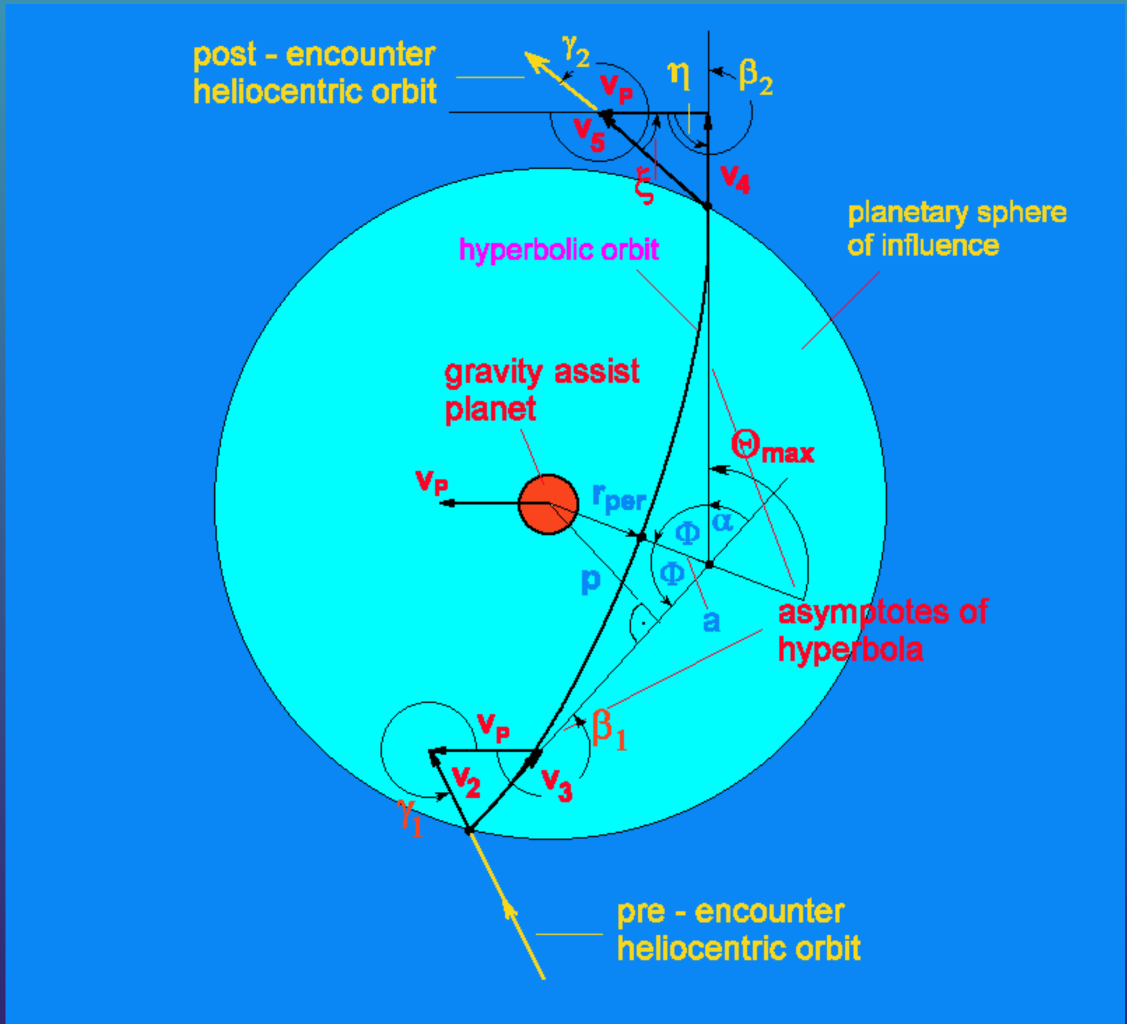
$$\mu = m/M << 1$$

Space Propulsion



$$M_P / r_{SOI}^2 \cong M_S / r_P^2 \rightarrow r_{SOI} \approx r_P \sqrt{M_P / M_S}$$

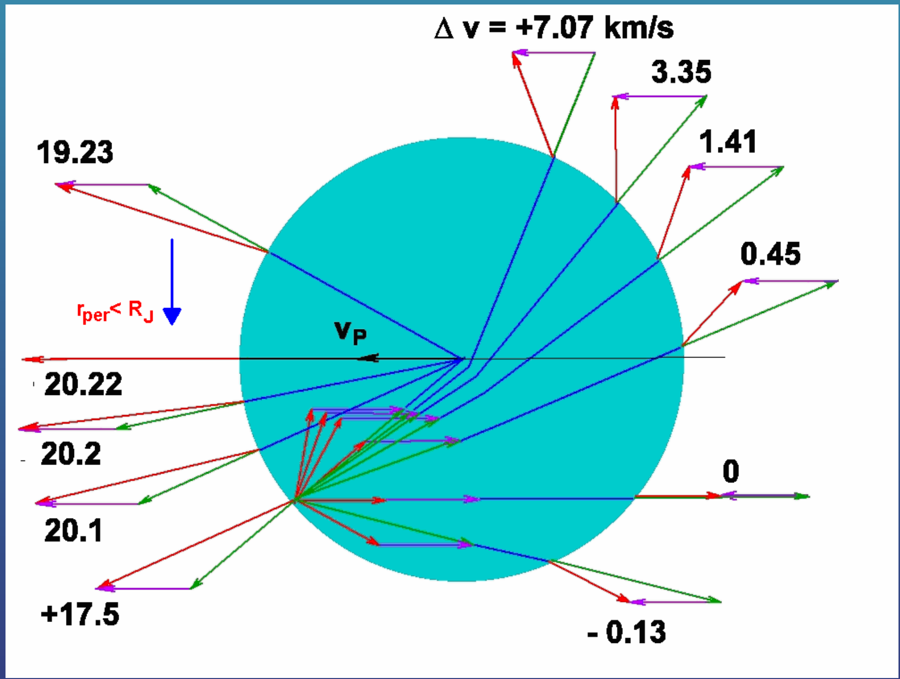
Space Propulsion

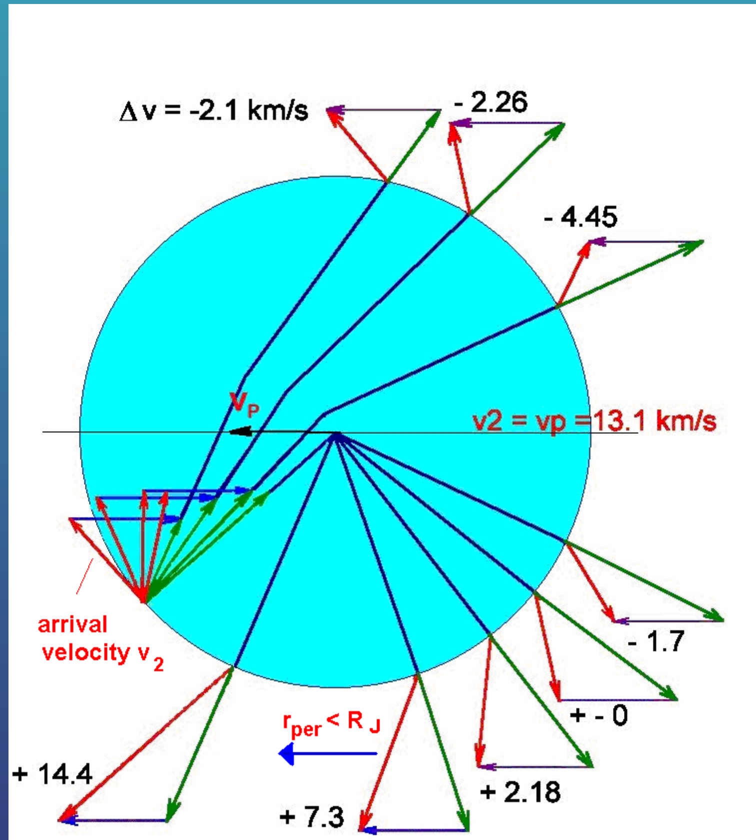


vectorial velocity addition at transfer between helocentric and planetocentric motions

Space Propulsion

Gravity assist at Jupiter
v_P = 13.1 km/s



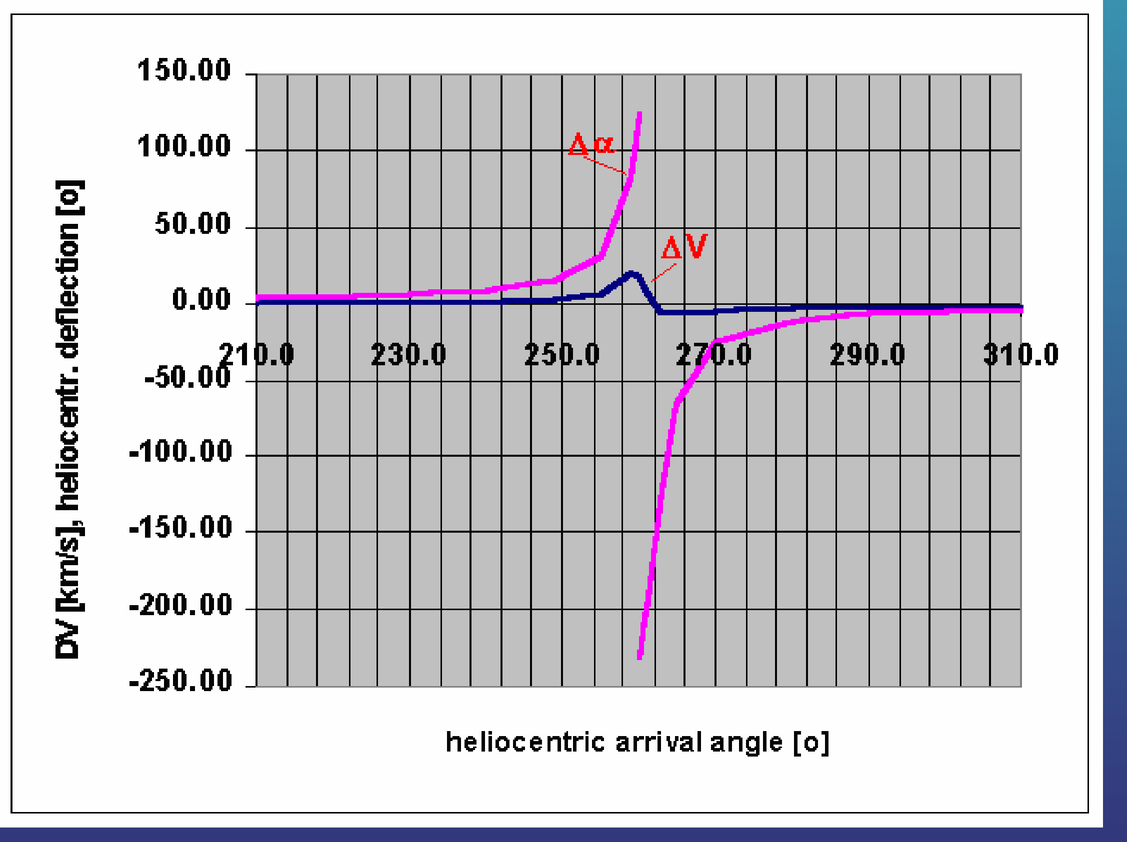


passage behind planet

passage in front of planet

gravity assist at Jupiter can boost S/C velocity to hyperbolic orbit so that it can leave solar system ($\Delta V > (2^{1/2}-1)\, v_P$ = 5.4 km/s)

Space Propulsion



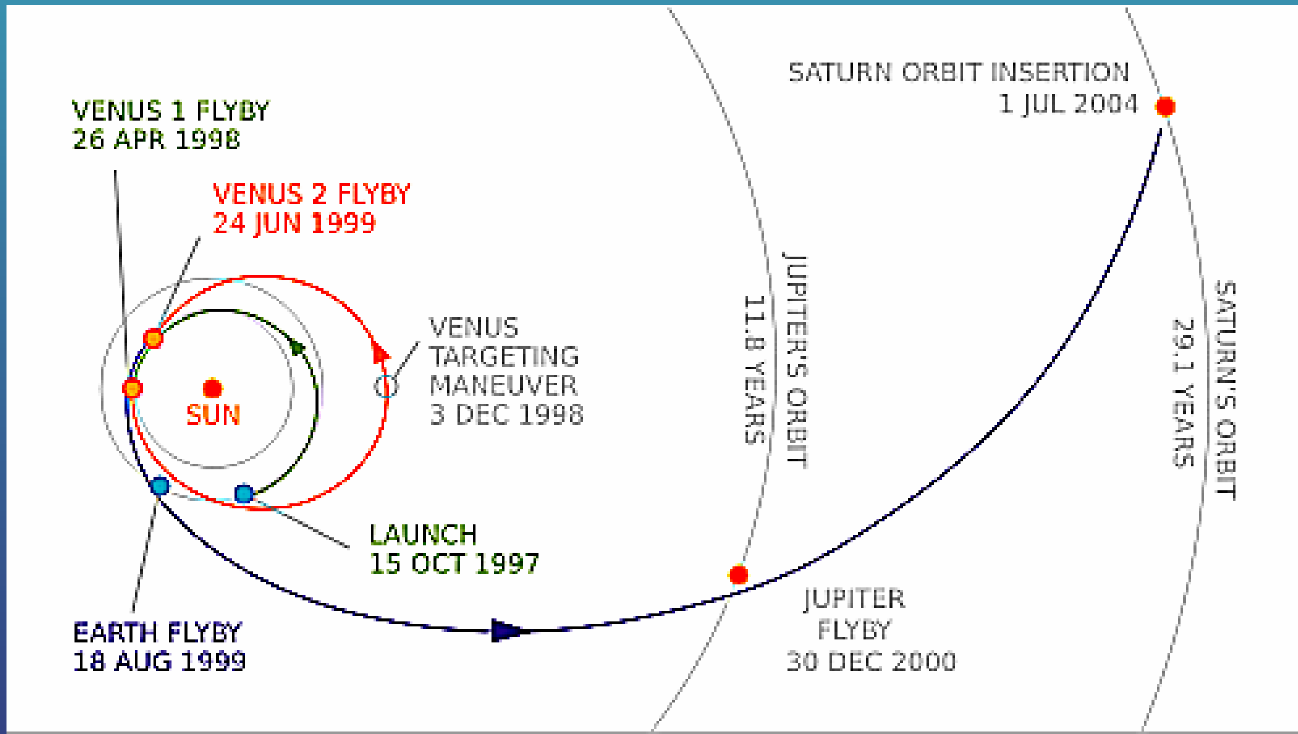
Space Propulsion

Maximum energy gain in gravity assist at different planets (closest approach = r_P)

	Planetary velocity [km/s]	Mass [kg]	Solar distance [10^6 km]	SOI radius [10^6 km]	Equatorial radius [km]	$v_{3,extr}$ [km/s]	ΔV [km/s]	
Mercury	47.87	3.28E23	57.9	0.024	2493	2.99	11.96	5
Venus	35.02	4.87E24	108.2	0.169	6051	7.33	16.03	2
Earth	29.78	5.97E24	149.6	0.259	6378	7.90	15.33	4
Mars	24.13	6.42E23	228.0	0.130	3394	3.55	9.27	8
Jupiter	13.06	1.90E27	778.4	24.05	71400	42.1	23.45	1
Saturn	9.65	5.69E26	1425.5	24.10	60000	25.2	15.59	3
Uranus	6.80	8.68E25	2870.4	18.96	25650	15.0	10.10	6
Neptune	5.43	1.03E26	4501.1	32.38	24780	16.7	9.54	7
Pluto	4.7	1.27E22	5900	0.47	1150	0.86	2.00	9

Space Propulsion

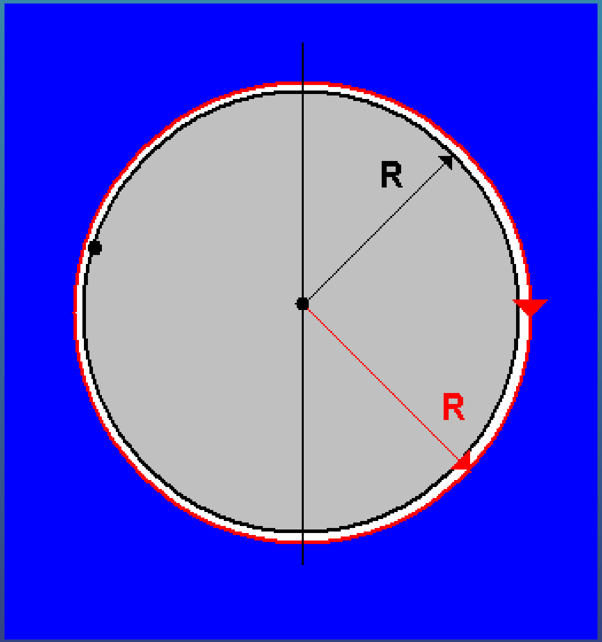
CASSINI probe to Saturn and Titan



	Hohmann transfer	gravity assists
total ΔV [km/s]	15.7	2
flight time [y]	6	6.7

Space Propulsion

Liftoff from ground



$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon = const.$$

energy conservation
ε = E/m … specific energy

$$\varepsilon_{gr} = \varepsilon_p = -\frac{\mu}{R}$$

specific energy at rest on Earth's surface
v = 0, r = R (purely potential energy)

$$V = \sqrt{\frac{\mu}{R}}$$

velocity in circular orbit with
Radius R

$$\varepsilon_{0,k} = \frac{\mu}{2R}$$

spec. kinetic energy in circular orbit
at Earth's surface, r =a = R

$$\varepsilon_O = \frac{\mu}{2R} - \frac{\mu}{R} = -\frac{\mu}{2R}$$

total specific energy in orbit near
Earth's surface

$$\Delta\varepsilon_O = -\frac{\mu}{2R} - \left(-\frac{\mu}{R}\right) = \frac{\mu}{2R}$$

spec. energy for liftoff from rest into
circular orbit near Earth's surface

$$\Delta V = \sqrt{2\Delta\varepsilon_O} = \sqrt{\frac{\mu}{R}} = \sqrt{\frac{3.98x10^5}{6.73x10^3}} \cong 7.69 \quad [km/s]$$

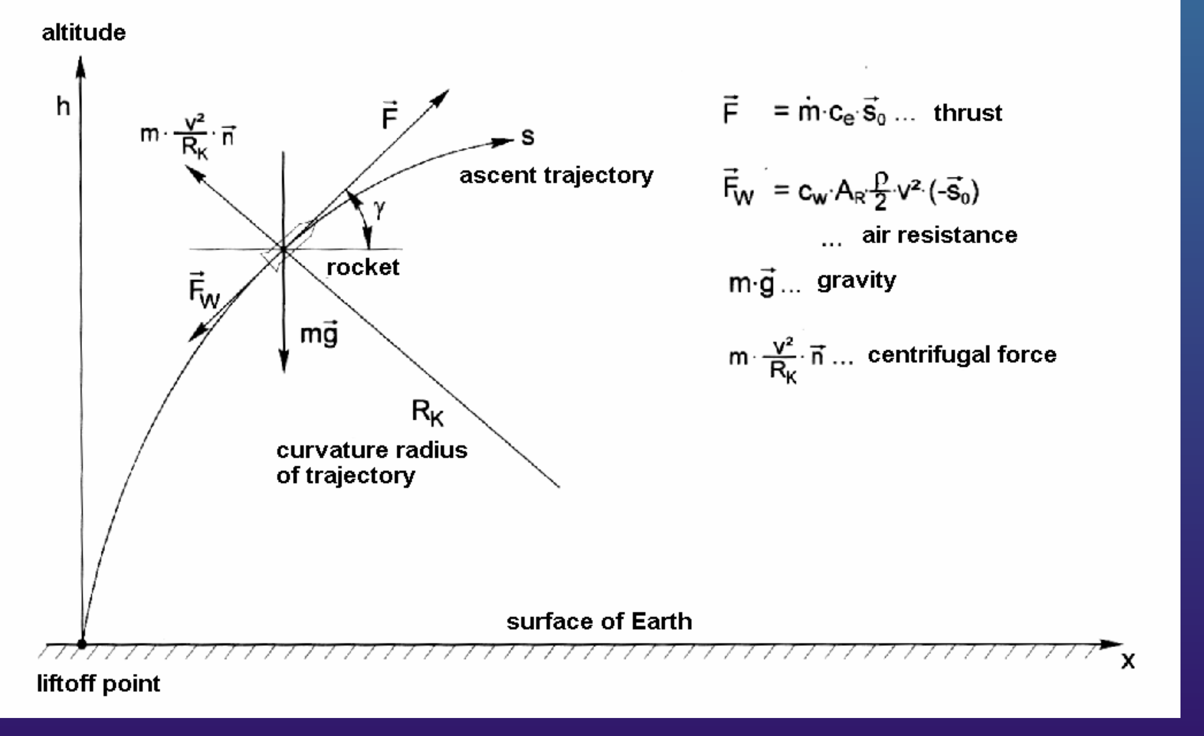
ΔV necessary for liftoff into
circular orbit near Earth's surface

Space Propulsion

Liftoff from ground

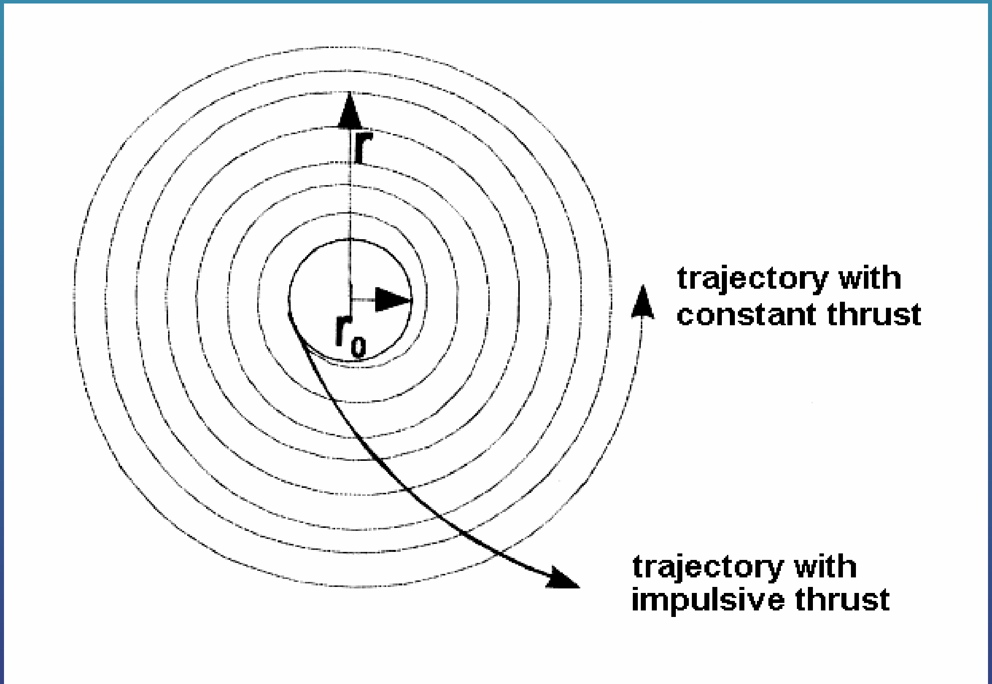
$$\Delta V = \sqrt{2\Delta\varepsilon_O} = \sqrt{\frac{\mu}{R}} = \sqrt{\frac{3.98x10^5}{6.73x10^3}} \cong 7.69 \quad [km/s]$$

$$m_p = m_f\left[\exp\left(\frac{\Delta V}{I_{sp}}\right) - 1\right] = 1\left\{\exp\left[\frac{(7690)}{(3100)}\right] - 1\right\} = 10.95 \quad [kg/kg]$$



Space Propulsion

Spiraling up



ΔV, propellant consumption and mission time can be estimated when

- thrust direction is always tangential to trajectory (permanent attitude change!!)
- thrust << gravity force

Space Propulsion

Spiraling up

$$\frac{d\vec{v}}{dt} = \frac{\vec{T}}{m} + \bar{g}$$

equation of motion for S/C of mass m, propelled by thrust T

$$\varepsilon = \frac{\bar{v}^2}{2} + U(\vec{r})$$

Specific energy = spec. kinetic energy + + potential

$$U(\vec{r}) = -\int_r^\infty \frac{F_G(\vec{r})}{m} d\vec{r} = -\int_r^\infty g(\vec{r}) d\vec{r} \rightarrow \frac{dU}{d\bar{r}} = -g(\vec{r})$$

potential energy and its gradient

$$\frac{d\varepsilon}{dt} = \frac{d}{dt}\left[\frac{\vec{v}^2}{2} + U(\vec{r})\right] = \vec{v}\frac{d\vec{v}}{dt} + \frac{dU(\vec{r})}{d\vec{r}}\frac{d\vec{r}}{dt} = \vec{v}\left(\frac{d\vec{v}}{dt} - \vec{g}(\vec{r})\right)$$

time derivative of specific energy

$$\frac{d\varepsilon}{dt} = \vec{v}\frac{\vec{T}}{m} = v\frac{T}{m}$$

acc. to assumption, always v II T; ➔ inner product replaced by magnitudes

Space Propulsion

Spiraling up

$$\frac{d\varepsilon}{dt} = \vec{v}\frac{\vec{T}}{m} = v\frac{T}{m}$$

at every moment, trajectories closely resemble circles (acc. to assumption T → 0)

$$v \cong \sqrt{\frac{\mu}{r}} \qquad \varepsilon \cong -\frac{\mu}{2r} \qquad \varepsilon = -\frac{\mu}{2a}$$

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon}{dr}\frac{dr}{dt} = +\frac{\mu}{2r^2}\frac{dr}{dt} = v\frac{T}{m} \approx \sqrt{\frac{\mu}{r}}\frac{T}{m}$$

$$\Delta V = \int_{t0}^{t}\frac{T}{m}dt = \int_{r0}^{r}\frac{\sqrt{\mu}}{2}\frac{dr}{r^{3/2}} = -\sqrt{\mu}.\frac{1}{\sqrt{r}}\Bigg|_{r0}^{r} = \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r}}$$

$$\Delta V = v_{c,0} - v_{c,r}$$

thrusting ΔV is equal to difference of velocities in initial and final orbit

Space Propulsion

Spiraling up

time required to spiral up from r_0 to r

assumption: T = const. → $\dot{m} = T / I_{sp}$ → dm/dt = const. $m = m_0 - \dot{m}(t - t_0)$

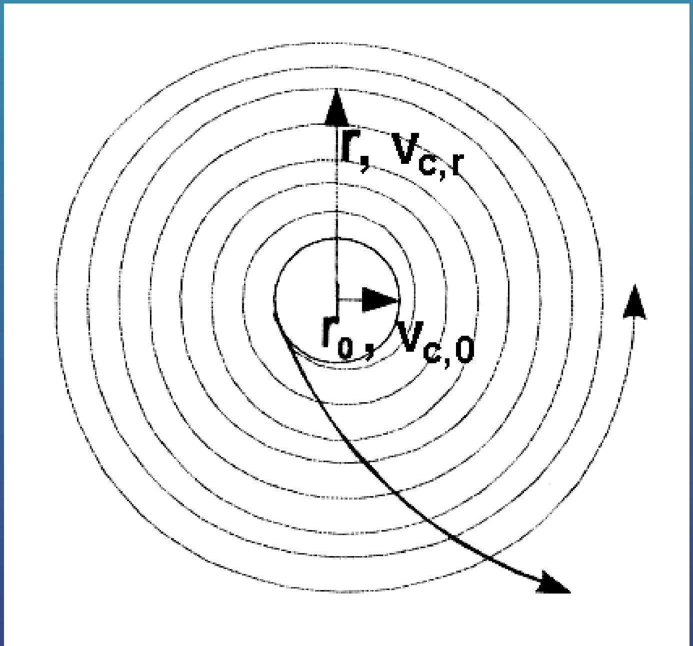
$$\Delta V = \int_{t0}^{t} \frac{T}{m} dt = \frac{T}{m_0} \int_{t0}^{t} \frac{dt}{1 - \frac{\dot{m}}{m_0}(t - t_0)} = -\frac{T}{\dot{m}} \ln\left[1 - \frac{\dot{m}}{m_0}(t - t_0)\right]$$

$$\Delta V = \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{r_0}}\left(1 - \sqrt{\frac{r_0}{r}}\right)$$

$$\tau = t - t_0 = \frac{m_0}{\dot{m}}\left\{1 - \exp\left[-\left(\sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r}}\right)\frac{\dot{m}}{T}\right]\right\} = m_0 \frac{I_{sp}}{T}\left[1 - e^{-(v_{c,0} - v_{c,1})/Isp}\right]$$

Space Propulsion

Spiraling up



$$\Delta V = v_{c,0} - v_{c,r}$$

$$\frac{\Delta V}{v_{c,0}} = 1 - \sqrt{\frac{r_0}{r}}$$

$I_{sp} >> \Delta V$

$$\tau = m_0 \frac{I_{sp}}{T}\left[1 - e^{-\Delta V / Isp}\right] = m_f \frac{I_{sp}}{T}\left(e^{\Delta V / Isp} - 1\right) \rightarrow m_f \frac{\Delta V}{T}$$

$$m_p = m_f \left[\exp\left(\frac{\Delta V}{I_{sp}}\right) - 1\right] \rightarrow m_f \frac{\Delta V}{I_{sp}}$$

Example

Electric propulsion: T = 5 mN, $I_{sp} = 10^5$ m/s;
Time and propellant mass required for spiraling up a 100 kg payload from 300 km LEO (v = 7730 m/s) to escape velocity?

$$m_\infty = 100.\left(e^{7809\sqrt{2}/10^5} - 1\right) \cong 11.7 \quad [kg]$$

$$\tau_\infty = 100 \frac{10^5}{5x10^{-3}}\left(e^{7809\sqrt{2}/10^5} - 1\right) \cong 5.1 \quad [y]$$

Space Propulsion

Comparison of Hohmann and spiral transfer

Hohmann

$$\Delta V_H = \left(\sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}\right) + \left(\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}\right)$$

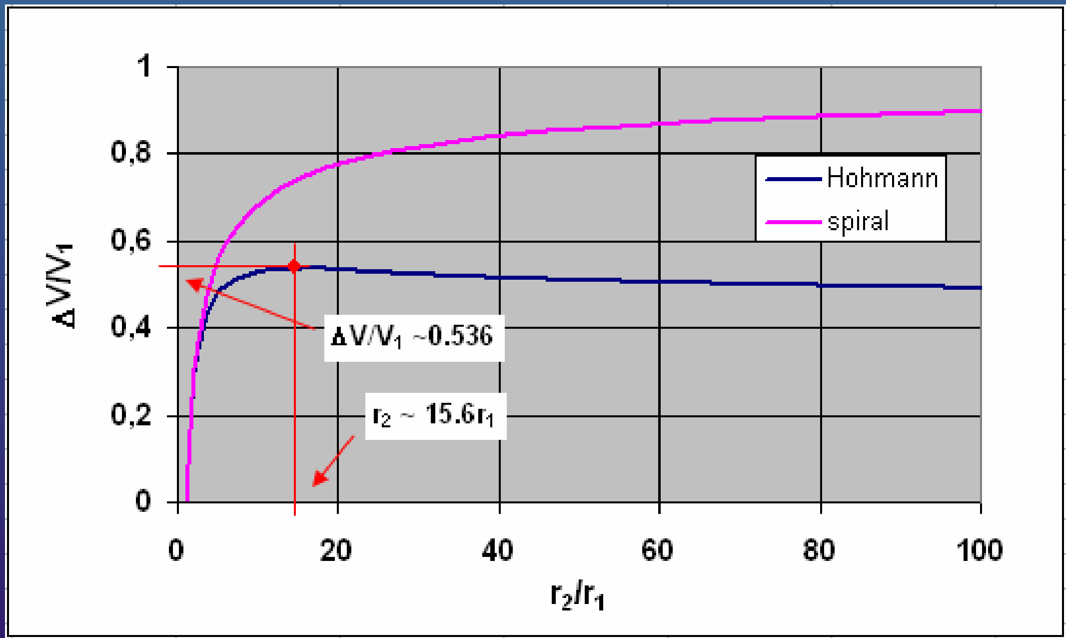
$$\frac{\Delta V_H}{V_1} = \left(\sqrt{\frac{2\rho}{1+\rho}} - 1\right) + \frac{1}{\sqrt{\rho}}\left(1 - \sqrt{\frac{2}{1+\rho}}\right)$$

spiral

$$\Delta V = V_1 - V_2$$

$$\frac{\Delta V}{V_1} = 1 - \sqrt{\frac{1}{\rho}}$$

$\rho = r_2/r_1$



Space Propulsion

Comparison of Hohmann and spiral transfer

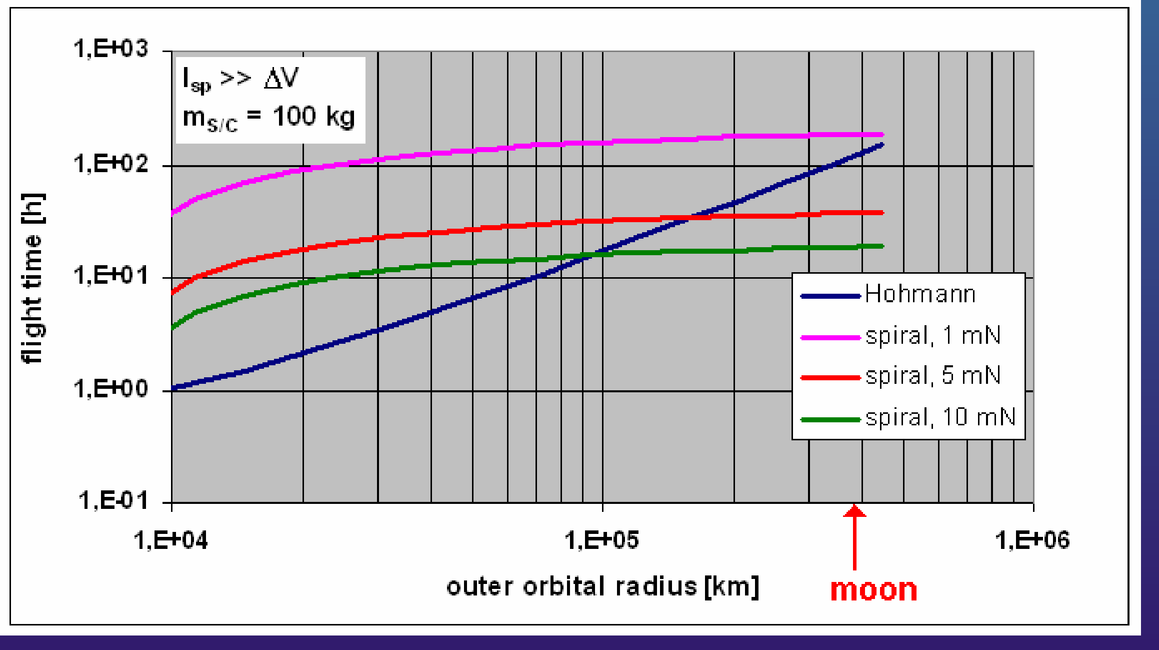
$$\tau_H = \frac{1}{2} P_{tr} = \pi \sqrt{\frac{a^3}{\mu}} = \pi \sqrt{\frac{r_1^3 (1+\rho)^3}{8\mu}}$$

depending **only** on orbital radii

$$\tau_{SP} \rightarrow m_f \frac{\Delta V}{T} = \frac{m_f}{T} \sqrt{\frac{\mu}{r_1}} \left(1 - \frac{1}{\sqrt{\rho}}\right)$$

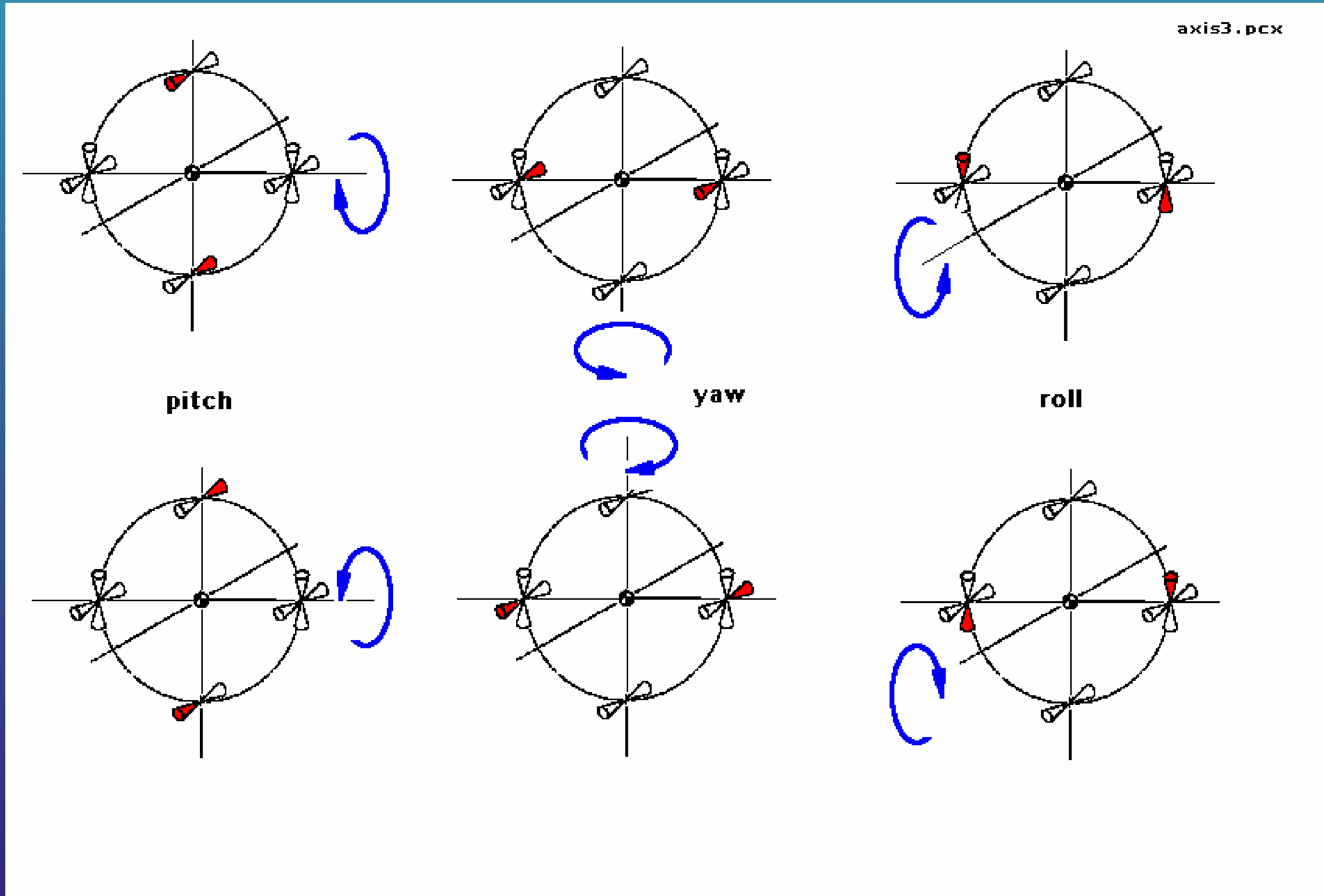
also depending on S/C mass, thrust (and specific impulse)

$$V_r = \frac{dr_2}{d\tau} = \frac{T}{m_f} \sqrt{\frac{r_1}{\mu}} . r_2$$



Space Propulsion

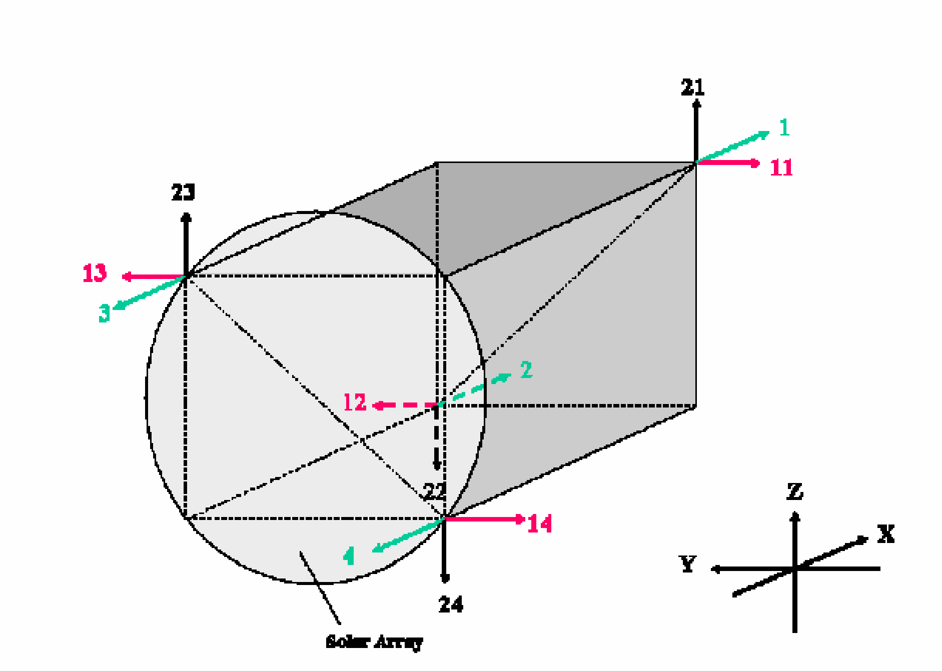
Attitude maneuvres



3 – axis controlled S/C

Space Propulsion

Thruster combinations to produce control forces and moments (HYPER, 2003) Option 1



Function	Primary Set	Secondary Set
Force in + X	3 + 4	(33 + 34)
Force in - X	1 + 2	(31 + 32)
Force in + Y	11 + 14	(31 + 34)
Force in - Y	12 +13	(32 + 33)
Force in + Z	22 + 24	(31 + 33)
Force in - Z	21 + 23	(32 + 34)
torque about X (+)	13 + 14	21 + 22
torque about X (-)	11 + 12	23 + 24
torque about Y (+)	2 + 3	21 + 24
torque about Y (-)	1 + 4	22 + 23
torque about Z (+)	2 + 4	11 + 13
torque about Z (-)	1 + 3	12 +14

Four clusters, each with 3 thrusters, are located at the corners of the S/C on opposite diagonals:

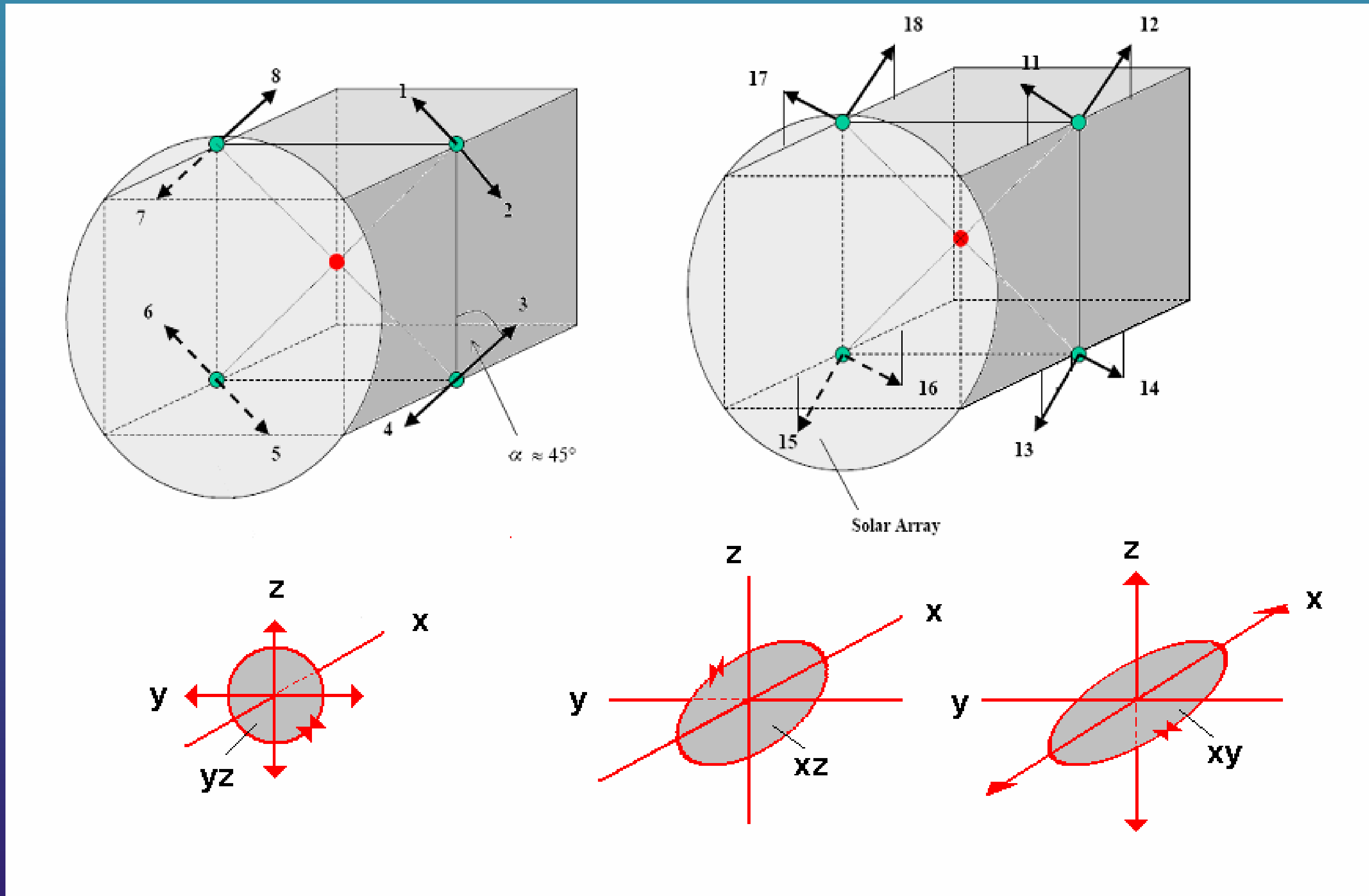
group {1, 2, 3, 4} is pointing into +/- X

group {11, 12, 13, 14} into +/- Y

group {21, 22, 23, 24} into +/- Z.

Space Propulsion

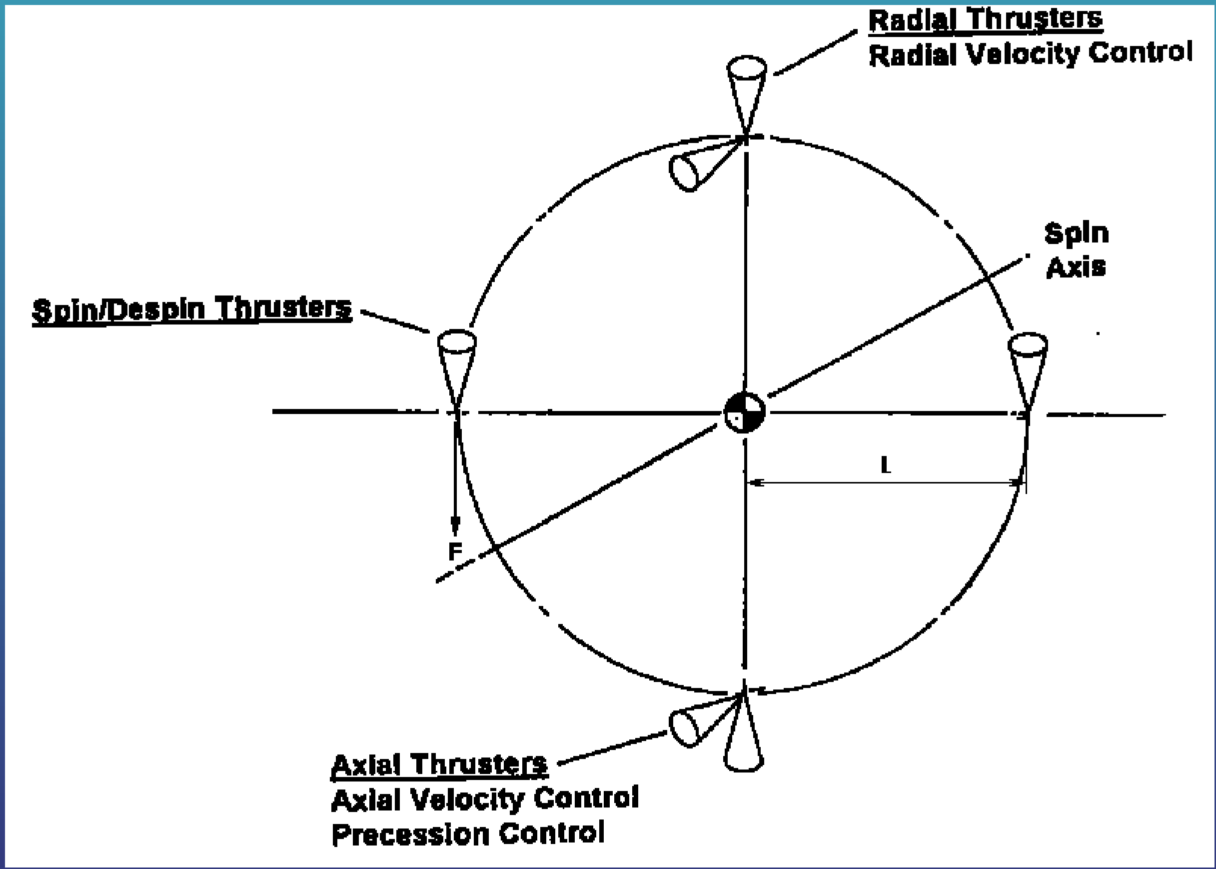
Thruster combinations to produce control forces and moments (HYPER, 2003) Option 2



Space Propulsion

Function	Primary Set	Secondary Set	Third Set
force in + X	11+ 13 + 15 + 17	11 + 15 + Torque x (-) pair, or 13 + 17 + Torque x (+) pair	
force in - X	12 + 14 +16 + 18	12 + 16 + Torque x (-) pair, or 14 + 18 + Torque x (+) pair	
force in + Y	2 + 3	5 + 8	2 + 3 + 5 + 8
force in - Y	6 + 7	1 + 4	1 + 4 + 6 + 7
force in + Z	4 + 5	2 + 7	4 + 5 + 2 + 7
force in - Z	1 + 8	3 + 6	1 + 8 + 3 + 6
torque about X (+)	1 + 5	3 + 7	
torque about X (-)	2 + 6	4 + 8	
torque about Y (+)	11 +14	16 + 17	
torque about Y (-)	12 + 13	15 + 18	
torque about Z (+)	11 + 13 + 16 + 18	11 + 18 + Force z (+) pair, or 13 + 16 + Force z (-) pair	
torque about Z (-)	12 + 14 + 15 + 17	12 + 17 + Force z (+) pair, or 14 + 15 + Force z (-) pair	

Space Propulsion



Attitude control thrusters on spin – stabilised S/C

Space Propulsion

Kinetics for rotational motion of S/C

rotational motion	Lin. analogon
T = torque [N.m]	F = force [N]
Θ = angle of rotation of the spacecraft [rad]	s = path [m]
ω = angular velocity of the spacecraft [rad/s]	v = velocity [m/s]
α = angular acceleration of the spacecraft during a firing, [rad/s^2]	a = acceleration [m/s^2]
I_v = mass moment of inertia of the vehicle, [kg.m^2]	m = mass [kg]
t_b = duration of the burn [s]	t = time [s]
H = change of spacecraft angular momentum during the firing, [kg.m^2/s]	p = momentum [m/s]

rotational motion	Lin. analogon
$\Theta = \frac{1}{2}\alpha t_b^2$	$s = \frac{a}{2}t^2$
$\alpha = \frac{T}{I_v}$	$a = \frac{F}{m}$
$\omega = \alpha t_b$	$v = at$
$H = I_v \omega$	$p = mv$
$H = T t_b$	$p = \int F dt \cong Ft$

Space Propulsion

Kinetics for rotational motion of S/C

Linear analogon

		Linear analogon
torque, produced by n thrusters, mounted at torque arm L, firing with equal thrust F	$T = nFL$	F
during the burn, the angular acceleration of the spacecraft is	$\alpha = \dfrac{nFL}{I_v}$	$a = \dfrac{F}{m}$
at shut down, the vehicle will have turned by	$\Theta = \dfrac{nFLt_b^2}{2I_v}$	$s = \dfrac{a}{2}t^2$
at shutdown, the spacecraft is left rotating at angular velocity	$\omega = \dfrac{nFL}{I_v}t_b$	$v = a.t$
angular momentum produced by a single firing is	$H = Tt_b$	$p = \int F dt$
propellant consumed during the burn is	$m_p = \dfrac{nFt_b}{I_{sp}} = \dfrac{H}{LI_{sp}}$	

$$I_{sp} = F / \dot{m}$$

Space Propulsion

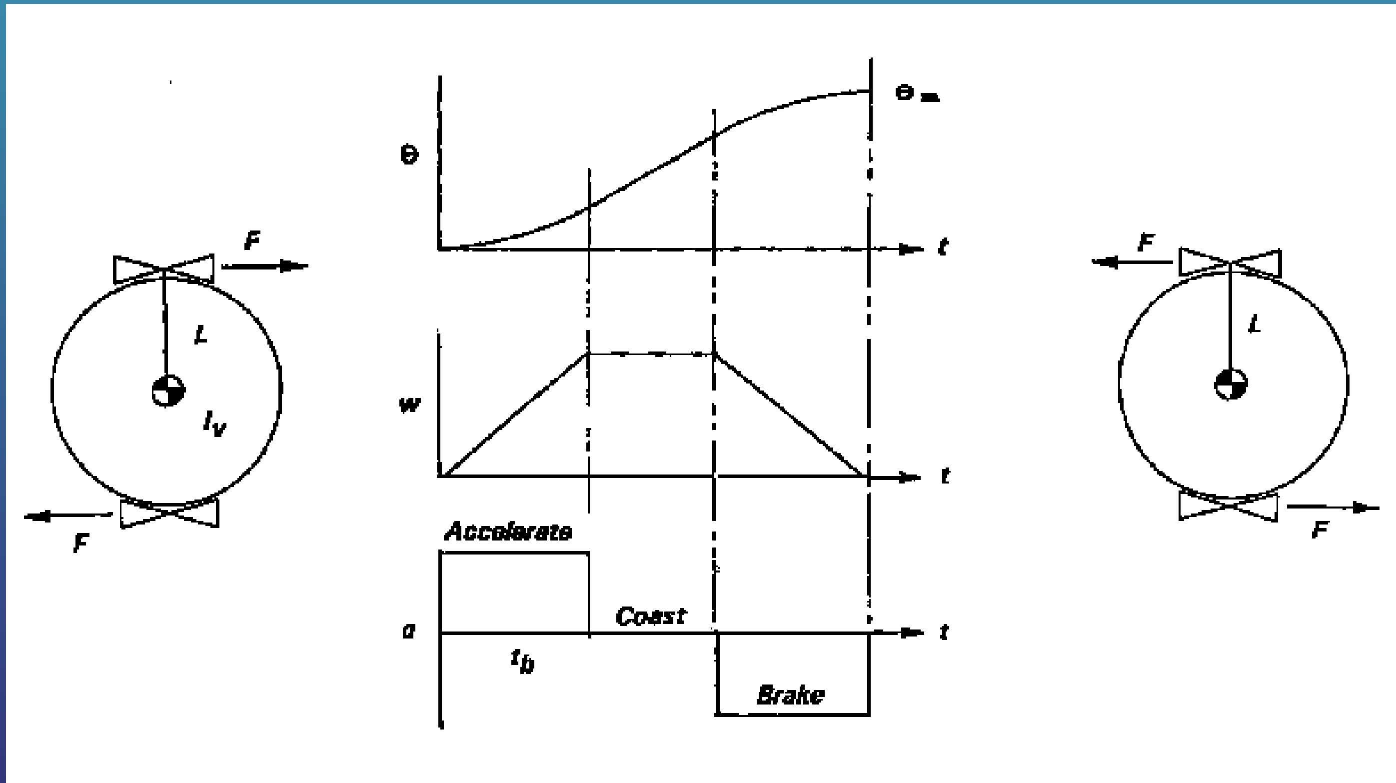
$$m_p = \frac{nFt_b}{I_{sp}} = \frac{H}{LI_{sp}}$$

shows the advantage of a long moment arm. The maximum moment arm is constrained in a surprising way: by the inside diameter of the launch vehicle payload fairing

Launch vehicle	Fairing i.d.[ft]
Atlas	9.6 or 12
Delta	8.3 or 10
Space Shuttle	15
Titan II	10
Titan III	13.1
Titan IV	16.7

Space Propulsion

one – axis maneuvre



Space Propulsion

total angle of rotation is

$\Theta_m = \Theta\,(\text{accelerating}) + \Theta\,(\text{coasting}) + \Theta\,(\text{braking})$

rotation during coasting is

$$\Theta = \omega t_c$$

the coasting rotation angle is

$$\Theta = \frac{nFL}{I_v} t_b t_c \quad \leftarrow \quad \omega = \frac{nFL}{I_v} t_b$$

total rotation during acceleration, coasting and braking is

$$\Theta_m = 2\left(\frac{nFL}{2I_v} t_b^2\right) + \frac{nFL}{I_v} t_b t_c = \frac{nFL}{I_v}\left(2t_b^2 + t_b t_c\right)$$

$$\Theta = \frac{nFLt_b^2}{2I_v}$$

accel.

maneuver time is

$t_m = t_C + 2t_b$

minimum rotation time is a fully powered maneuver with zero coast time

$$t_{\min} = 2t_b = \sqrt{\frac{2.\Theta_m I_v}{nFL}}$$

thrust level required for each thruster at given minimum rotation time

$$F = 2\frac{\Theta_m I_v}{nLt_{\min}^2}$$

propellant required for a one-axis maneuver is twice the single burn consumption

$$m_p = 2\frac{nFt_b}{I_{sp}} = \frac{nFt_{\min}}{I_{sp}}$$

$$m_p = \frac{nFt_b}{I_{sp}} = \frac{H}{LI_{sp}}$$

Space Propulsion

Example 6: One-Axis Maneuver

Find the minimum time required for a spacecraft to perform a 90-deg turn about the z axis with two thrusters if the spacecraft has the following characteristics:

Mass of S/C = 500 kg,

Radius of S/C = 0.75 m

→ Moment of inertia about the z axis $\cong (2/5) M_{S/C.} L^2 = 112.5\ kg.m^2$

Moment arm = 0.75 m

Thrust of each engine = 10 N

and

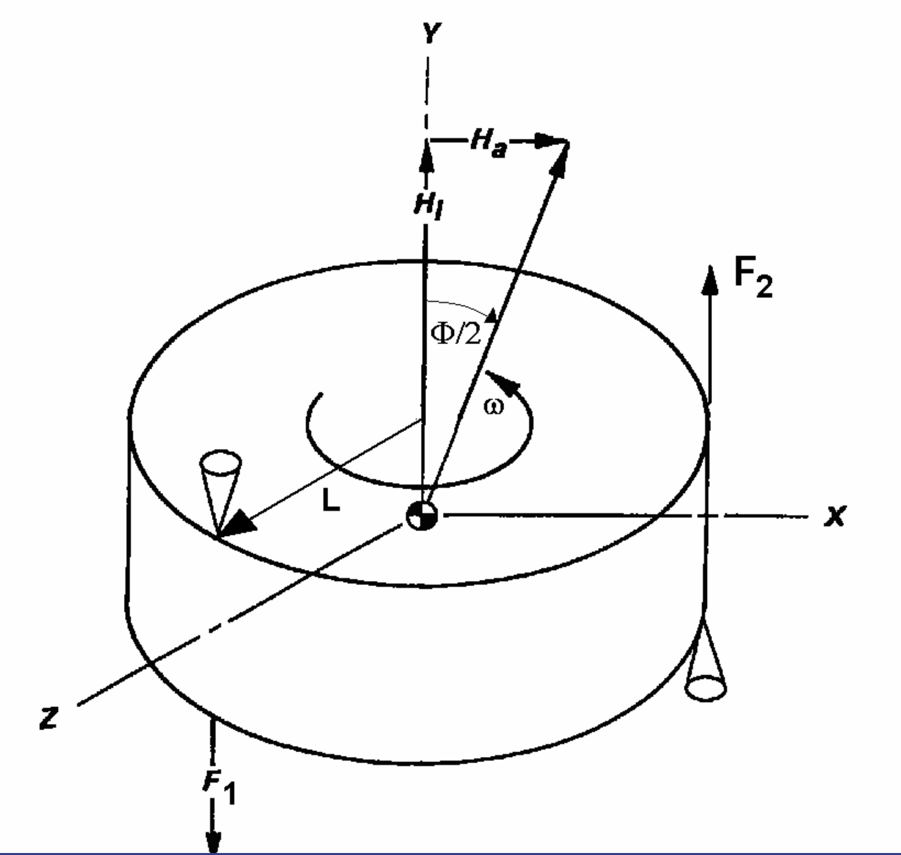
$\Theta_m = \pi/2 = 1.5708\ rad$

$$t_{\min} = \sqrt{\frac{2.\Theta_m I_v}{nFL}} = \sqrt{\frac{2*1.5708*112.5}{2*10*0.75}} = 4.854 \quad s$$

How much propellant was consumed by the maneuver if $I_{sp} = 1900\ m/s$ *?*

$$m_p = 2\frac{nFt_m}{I_{sp}} = \frac{2*2*10*4.854}{1900} = 0.102 \quad kg$$

Space Propulsion



precession of spin axis

H_i … initial angular momentum
H_a … applied angular momentum

$$\Phi / 2 \approx \frac{H_a}{H_i} = \frac{nFLt_b}{I_y \omega}$$

nutation angle caused by application of single thrust pulse

***Two pulses** are required to precess the spin axis; both pulses are parallel to the spin axis. After the First pulse, the spin axis will continue to precess until a second pulse of equal magnitude and opposite direction is fired. The spin axis can be repositioned by selecting the timing of the second pulse.*
*The first pulse is used to **cause nutation** at an angle of **one-half** the desired precession. The second pulse **stops the nutation** and provides the **remaining half** of the desired angle*

Space Propulsion

Example 7: Precession of Spin Axis

What burn time, or pulse width, is required to precess a spacecraft spin axis by 3-deg (0.05236 rad) under the following conditions:

Thrust 10 N
Moment arm = 0.5 m
Moment of inertia 112.5 kg.m2
Spacecraft Spin rate 2 rpm (0.2094 rad/s)
Specific impulse = 1900 m/s

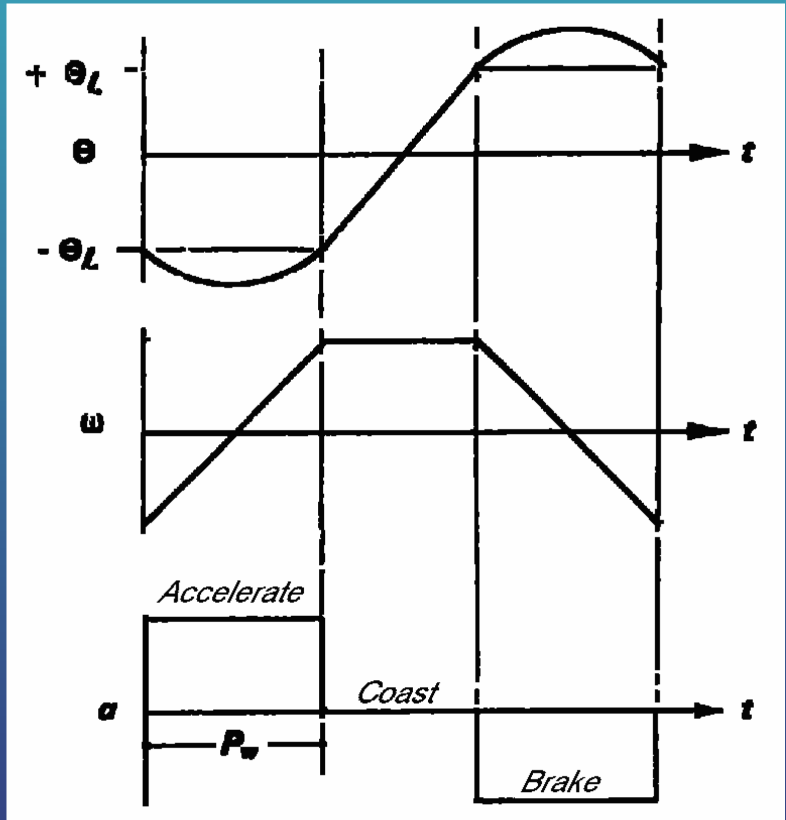
$$t_b = \frac{\Phi I_v \omega}{2nFL} = \frac{0.05236 * 112.5 * 0.2094}{2*1*10*0.5} = 0.124 \quad [s]$$

burn time of thruster to produce nutation angle $\Phi/2$

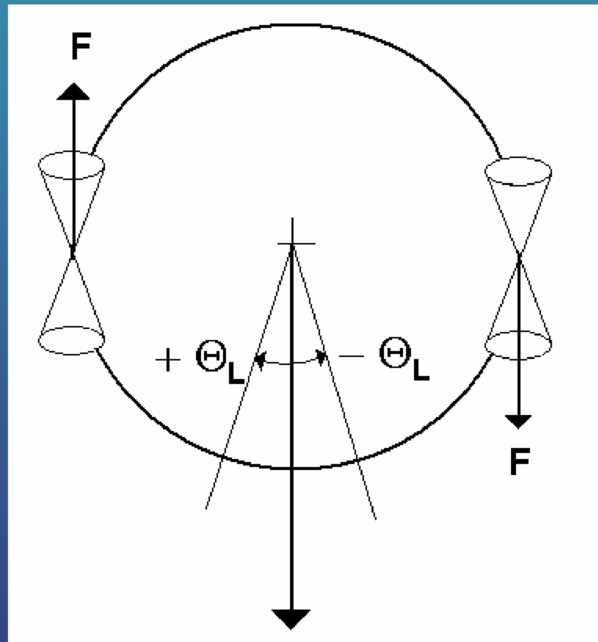
$$m_p = 2\frac{nFt_b}{I_{sp}} = \frac{2*1*10*0.124}{1900} = 0.0013 \quad [kg] = 1.3 \quad [g]$$

total propellant consumed by both burns

Space Propulsion



limit cycle without external torque



A limit cycle without external torque swings the spacecraft back and forth between preset angular limits. When the spacecraft drifts across one of the angular limits Θ_L , the attitude-control system fires a thruster pair for correction. The spacecraft rotation reverses and continues until the opposite angular limit is reached, at which time the opposite thruster pair is fired. It is important that the smallest possible impulse be used for the corrections because the impulse must be removed by the opposite thruster pair.

cont'd

Space Propulsion

$$\Theta = \frac{nFL}{I_v} t_b t_c$$

$$\Theta_{tot} = \frac{nFL}{I_v}\left(\frac{P_w^2}{2} + \frac{t_c P_w}{2}\right)$$

total angle of rotation Θ ↓ replacing $2t_b \rightarrow P_w$

(EQ 86)

$$\Theta_L = \frac{1}{2}\frac{nFL}{I_v} t_b t_c = \frac{nFL}{4I_v} P_w t_c$$

the limit settings $\pm\ \Theta_L$ are one-half of the coasting angle ↓
$\Theta = 2\ \Theta_L$ (neglecting small rotations during accel & brake)

$$m_{p,cyc} = 2\frac{nFP_w}{I_{sp}}$$

each cycle includes two pulses;
the propellant consumed per cycle is

Propellant consumption is small for low thrust, short burn time, and high specific impulse in pulsing operation. Pulsing engines are characterized by **minimum impulse bit I_{min}**

$$I_{min} = (F.P_w)_{min}$$

The minimum impulse bit is a characteristic of a given thruster/valve combination

cont'd

Space Propulsion

pulsing properties of attitude – control thrusters

	Min thrust [mN]	Min impulse bit [mN.s]	Pulsing I_{sp} [m/s]
Cold-gas -Helium	50	5 - 10	800
Cold-gas-Nitrogen	50	5 - 10	500
Monopropellant - N_2H_4	500	50 - 100	1200
Bipropellant - N_2O_4/MMH	10000	750 - 1500	1200

$$\omega = \frac{nFL}{I_v} t_b \quad \rightarrow \quad t_c = \frac{4 I_v \Theta_L}{nFLP_w}$$

coast time through $2\Theta_L$

$$t_{cy} = t_c + 2P_W = \frac{4 I_v \Theta_L}{nLI_{\min}} + 2P_w$$

length of a cycle (from $+\Theta_L$ to $-\Theta_L$) if minimum impulse bits are used; *usually, P_W can be neglected*

$$m_{p,cyc} = 2\frac{nFP_w}{I_{sp}} \quad \rightarrow \quad \dot{m}_p = \frac{m_{p,cyc}}{t_{cy}} \sim \frac{n^2 I_{\min}^2 L}{2 I_{sp} I_v \Theta_L}$$

propellant consumption per unit time

Example 8: Limit-Cycle Operation

A spacecraft with 112.5 kg.m² inertia uses 5N thruster pairs mounted at a radius of 0.5 m from the center of mass. For limit-cycle control to Θ_L = 0.5 deg (0.008727 rad), what is the propellant consumption rate if I_{sp} is 1900 m/s, the pulse duration is 30 ms, and there are no external torques. ?

$$t_{cy} = 2P_w + \frac{4I_v\Theta_L}{nFLP_w} = 0.06 + \frac{4*112{,}5*0{,}008727}{2*5*0{,}5*0{,}030} = 0{,}06 + 26{,}181 = 26{,}241 \ \ [s]$$

time for 1 cycle

$$m_{p,cy} = 2\frac{nFP_w}{I_{sp}} = 2\frac{2*5*0.03}{1900} = 0.00032 \ \ [kg/cycle] = 0{,}32 \ \ [g/cycle]$$

propellant consumed per cycle

$$\dot{m}_p = \frac{m_{p,cy}}{t_{cy}} = \frac{0{,}00032}{26{,}241} \cong 1.2x10^{-5} \ \ [kg/s] = 1.037 \ \ [kg/day]$$

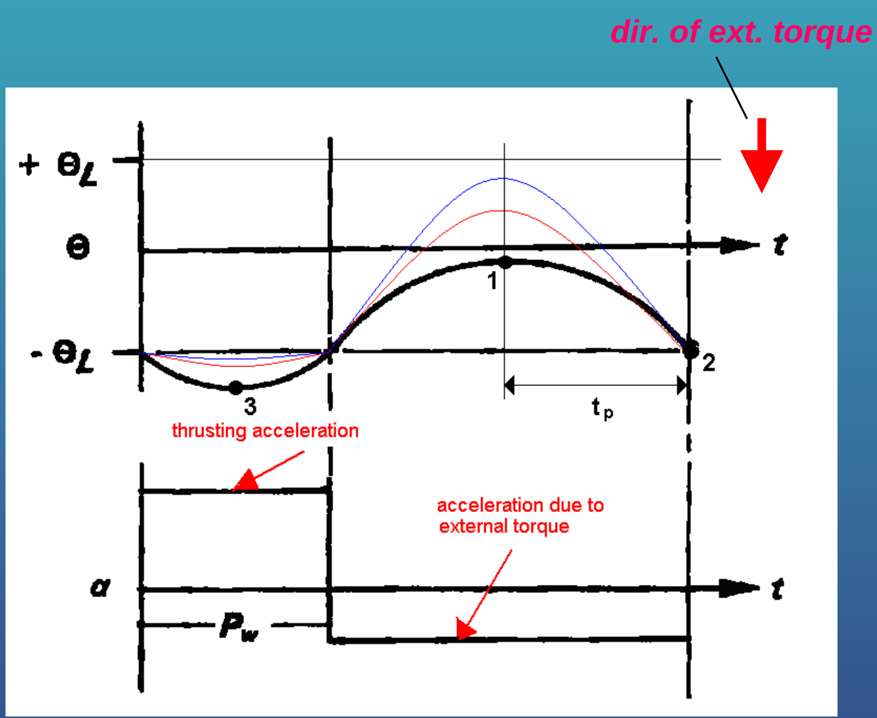
average propellant consumption rate

Space Propulsion

Simplified equations for external torques

Disturbance	Type	Influenced primarily by	Formula
Gravity gradient	Constant or cyclic, depending on vehicle orientation	Spacecraft geometry, Orbit altitude	$T_g = \frac{3\mu}{r^3}\lvert I_z - I_y \rvert \Theta$ ~ $4\mathrm{x}10^{-5}$ [Nm] where T_g is the max gravity torque; μ is the Earth's gravity constant (398,600 *km³/s²*), *r* the orbit radius, *Θ* the max deviation of the *z* axis from vertical in radians; I_z and I_y are moments of inertia about z and y (or x, if smaller) axes.
Solar radiation	Constant force but cyclic on Earth-oriented vehicles	Spacecraft geometry, Spacecraft surface area	The worst-case solar radiation torque $T_{sp} = P_s A_s L_s (1+q)\cos i$ ~ $7\mathrm{x}10^{-7}$ [Nm] is due to a specularly reflective surface, where P_S is the solar constant, 4.617 x 10^{-6} N/m²; A_s is the area of the surface, L_s the center of pressure to center of mass offset, i the angle of incidence of the sun, and q the reflectance factor that ranges from 0 to 1; q = 0.6 is a good estimate
Magnetic field	Cyclic	Orbit altitude, Residual spacecraft magnetic dipole, Orbit inclination	$T_m = DB$ where T_m is the magnetic torque on the spacecraft, *D* the residual dipole moment of the vehicle in A.m², and B the Earth's magnetic field in Tesla. *B* can he approximated as $2M/r^3$ for a polar orbit to half that at the equator. M is the magnetic moment, 8 x 10^{25} emu at Earth, and r is radius from dipole (Earth) center to spacecraft in centimeters
Aerodynamic	Constant for Earth-oriented vehicle in circular orbit	Orbit altitude, Spacecraft configuration	$T_a = \sum F_i L_i$ T_a is the summation of the forces F_i on each of the exposed surface areas times the moment arm L_i to the center of each surface to the center of mass, where $F = \rho C_d A V^2/2$ with F the force, C_d the drag coefficient (usually between 2.0 and 2.5), p the atmospheric density, A the surface area, and V the spacecraft velocity.

Space Propulsion



one-sided limit cycle

with an external torque on the spacecraft, rotation occurs until a limit line is reached and a thruster pair is fired for correction

total angular momentum H supplied by the propulsion system exactly equals the momentum induced by the external torque T_x during mission time t_m; $\overline{F}$ is time – averaged thrust

$$H = T_x t_m = \overline{F} t_m$$

propellant mass required to compensate for the external torque

$$m_p = \frac{n\overline{F}}{I_{sp}} t_m = \frac{n\overline{F}L}{LI_{sp}} t_m = \frac{T_x}{LI_{sp}} t_m$$

cont'd

Space Propulsion

S/C rotation is accelerated by from zero speed at the extreme limit $+\Theta_L$ (point 1) through an angular path of < 2ΘL with an angular acceleration α_x, generated by the **external torque** only. The opposite limit angle will be reached after an angular interval $2\Theta_L$ and a "pass" time t_p (approximately equal to half the cycle time t_{cy}).

$$2\Theta_L = \frac{1}{2}\alpha_x t_p^2$$

$$t_{cy} =\sim 2t_p = 4\sqrt{\frac{\Theta_L}{\alpha_x}} = 4\sqrt{\frac{\Theta_L I_v}{T_x}}$$

angular speed ω_L, at the end of the cycle, at $-\Theta_L$ (at point 2) is

$$\omega_L = \alpha_x t_{cy}/2 = 2\sqrt{\frac{T_x \Theta_L}{I_v}}$$

Now the thrusters are firing, producing a thrusting angular acceleration α. They reduce this angular speed to 0 (at the turning point 3) after a burning time of $P_W/2$

$$\omega_L = \alpha . P_W / 2 = \frac{nLFP_W}{2I_v}$$

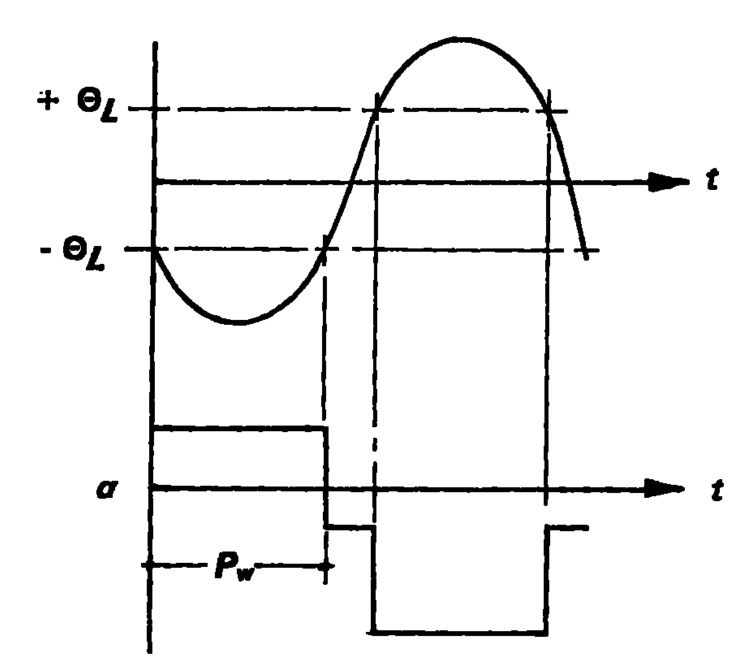
From that follows the ***impulse per thruster***, required to turn around the angular speed of the S/C, so that it moves against external torque up to an angle of not larger than Θ_L

$$FP_W \leq \frac{4}{nL}\sqrt{T_x I_v \Theta_L}$$

If minimum impulse bits I_{min} are used, the rotation limit must be wider than a minimum Θ_L, in order to avoid thrusters being fired in the direction of external torque. This would cause excessive propellant consumption.

$$\Theta_L > \frac{n^2 L^2 I_{min}^2}{16 I_v T_x}$$

Space Propulsion



forced limit cycle

A forced limit cycle occurs when thrusters are fired in the direction of the external torque; that is, when the condition

$$\Theta_L > \frac{n^2 L^2 I_{min}^2}{16 I_v T_x}$$

is not met

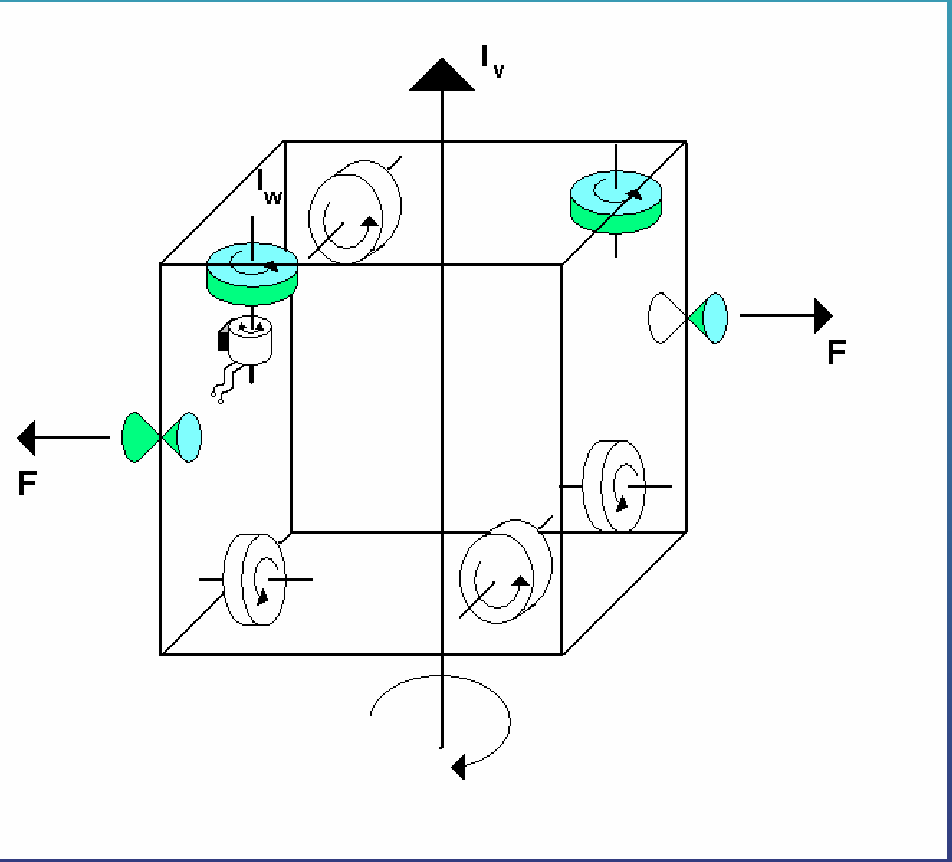
propellant consumed in a forced limit cycle is

$$m_p = \frac{I_v R^2}{L \Theta_L I_{sp}} t_m$$

R [Hz] = $1/t_{cy}$ = limit-cycle rate of the system
t_m [s] = mission duration

R can only be calculated numerically from a higher order equation containing the parameters I_{min}, P_W, T_x, I_v, L, Θ_L.

Space Propulsion



reaction wheel maneuvres

To perform a rotational maneuver with a reaction wheel, the flywheel is accelerated by a motor. The spacecraft accelerates in the opposite direction.

cont'd

Space Propulsion

A S/C can be rotated by an angle Θ by application of a torque T for time interval t

$$\Theta = \frac{Tt^2}{2I_v}$$

this torque can be supplied by an accelerating flywheel; angular acceleration α_W is supplied by a motor

$$T = \alpha_W I_W$$

The resulting S/C rotation angle is

$$\Theta = \frac{\alpha_W I_W t^2}{2I_v}$$

and the increase in wheel speed:

$$\Delta\omega_w = \alpha_w t$$

The S/C can be returned to its original position by applying the opposite torque to the flywheel; the net increase in flywheel rotational speed then is 0 (neglecting friction). Due to unbalanced torques however, the flywheel eventually will reach its upper angular speed limit and then is not fully available for maneuvering any more. To become maneuverable again it must be „**unloaded**", i.e. its angular speed must be brought to 0 again.

cont'd

Space Propulsion

total angular momentum of a fully loaded wheel is

$$H = I_w \omega_{w,\max}$$

to unolad the wheel, a torque in the opposite direction must be applied to it by the motor for a certain time; in order not to produce net rotation of the S/C, an equal and opposite momentum must be supplied by the thrusters:

$$H = Tt = nFLt$$

time required for unloading is

$$t = \frac{H}{nFL} = \frac{I_w \omega_{w,\max}}{nFL}$$

propellant consumption for unloading is

$$m_p = \frac{nF.t}{I_{sp}} . \frac{L}{L} = \frac{I_w \omega_w}{L I_{sp}}$$

cont'd

Space Propulsion

Example 9: Reaction Wheel Unloading

How much propellant does it take to unload one of the Magellan wheels, and how long does it take? (JPL Venus Mission, 1994)

The Magellan wheel characteristics are:
maximum momentum = 27 N-m-s *maximum wheel speed = 4000 rpm = 418.879 rad/s*

The thruster pair to be used has the following characteristics:
thrust = 1 N; moment arm = 2m *pulsing specific impulse 1500 m/s.*

the propellant mass required to unload it is

$$m_p = \frac{H}{LI_{sp}} = \frac{27}{2*1500} = 0.009 \quad [kg]$$

engine burn time required to unload is

$$t = \frac{H}{nFL} = \frac{27}{2*1*2} = 6.75 \quad [s]$$